ALIGNED WITH
2021 REVISION

LEARNING MATERIAL

SIGNALS & SYSTEMS

GOVERNMENT OF KERALA

DEPARTMENT OF TECHNICAL EDUCATION

STATE INSTITUTE OF TEACHERS' TRAINING & RESEARCH (SITTTR)

Kalamassery

REVISION 2021

SIGNALS & SYSTEMS

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PREFACE

Welcome to the world of Signals and Systems! This learning material has been carefully crafted to provide you with a comprehensive understanding of this fundamental field, specifically tailored for students pursuing a diploma program. Whether you are embarking on a career in engineering, telecommunications, or any other field that deals with the analysis and processing of signals, this material will serve as your guide.

Signals and Systems is an essential subject that forms the backbone of modern technology. It encompasses the study of how information is captured, manipulated, and transmitted in various systems. From audio and image processing to telecommunications and control systems, Signals and Systems underlies the design and analysis of a wide range of technologies that shape our world.

Signals and Systems is an Engineering Science course in Revision 2021 diploma syllabus. This course is in the fifth semester for Electronics & Communication Engineering program as a program core course and in the sixth semester Biomedical Engineering program as a program elective course. The course material is prepared in accordance with the syllabus prescribed by the Department of Technical Education, Government of Kerala and State Institute of Technical Teachers' Training & Research (SITTTR).

This learning material aims to present the key concepts, theories, and techniques in a clear and concise manner, allowing you to grasp the fundamental principles of Signals and Systems. Each chapter builds upon the previous ones, providing a structured learning experience that gradually increases in complexity. The material includes numerous examples, illustrations, and practical applications to enhance your understanding and demonstrate the real-world relevance of the subject matter.

This material is divided into four modules. Each module covers a course outcome. The course material is prepared in such a way that the students find it easy to understand the concepts and get the basic idea of mathematical representation and analysis of signals and systems. The faculty members who prepared the material have taken great effort to make the topics simple even for the beginners. At the end of each module, sample questions are included which will help students to prepare for the board examination.

Suggestions are invited to make it better in the revised editions. We hope this material will benefit both students and teachers.

CONTRIBUTORS

ACKNOWLEDGEMENT

We would like to extend our heartfelt gratitude and appreciation to Ms. Geethadevi R., Joint Director in-charge and Ms. Chandrakantha, Deputy Director (Rtd.) for their visionary leadership and guidance throughout the development of this learning material. Their valuable insights and encouragement have been instrumental in shaping this learning material and ensuring its relevance to the diploma program.

We would also like to acknowledge the diligent efforts of Dr. Ajitha S. and Ms. Swapna K. K., Project Officers, SITTTR Kalamassery who have worked tirelessly behind the scenes. Their expertise, organizational skills, and attention to detail have been invaluable in coordinating the various aspects of this endeavour, from content development to editing and formatting. Their commitment to excellence has played a pivotal role in ensuring the quality of this learning material.

Furthermore, we extend our appreciation to the team of subject matter experts, curriculum designers, and educators who have generously shared their knowledge and expertise. Their contributions have helped to create a comprehensive and well-rounded resource that caters to the learning needs of the students in the diploma program.

We are grateful to all the reviewers who provided valuable feedback and suggestions during the development stages. Their critical insights and constructive criticism have greatly enhanced the clarity and effectiveness of the material.

Last but not least, we would like to acknowledge the students who have been our ultimate motivation. Your eagerness to learn and succeed has been the driving force behind this endeavour. It is our hope that this learning material will serve as a valuable resource to enhance your understanding and mastery of the subject.

To all those mentioned above, and to anyone who has contributed in ways both seen and unseen, we extend our deepest appreciation. Your dedication, expertise, and unwavering support have been instrumental in the creation of this learning material, and we are truly grateful for your invaluable contributions.

With heartfelt thanks,

Sheeja Rajan S.

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CONTENTS

Program : **Diploma in Electronics and Communication/ Biomedical Engineering**	
Course Code :	Course Title: **Signals and Systems**
Semester : **5/6**	Credits: **4**
Course Category: **Program Core/ Elective**	
Periods per week: **4 (L:3, T:1, P:0)**	Periods per semester: **60**

Course Objectives:

- To introduce students to the idea of signals and systems, their characteristics in time and frequency domain.
- To provide basic knowledge on Fourier representation and Laplace transform and its applications on signals and systems

Course Prerequisites:

Topic	Course code	Course name	Semester
Differentiation, Integration	1002 2002	Mathematics I & II	1 & 2

Course Outcomes:

On completion of the course, the student will be able to:

COn	Description	Duration (Hours)	Cognitive level
CO1	Summarize the basic concepts, classifications and mathematical properties of signals	14	Understanding
CO2	Classify and compare continuous time and discrete time systems	12	Understanding
CO3	Explain Fourier representation of signals	16	Understanding
CO4	Apply Laplace transform to demonstrate the concepts of signals and systems	16	Applying
	Series Test	2	

CO – PO Mapping:

Course Outcomes	PO1	PO2	PO3	PO4	PO5	PO6	PO7
CO1	2						
CO2	2						
CO3	2						
CO4	3						

3-Strongly mapped, 2-Moderately mapped, 1-Weakly mapped

Course Outline:

Module Outcomes	Description	Duration (Hours)	Cognitive Level
CO1	**Summarize the basic concepts, classifications and mathematical properties of signals**		
M1.01	Define signals and its importance.	3	Understanding
M1.02	Explain the basic elementary signals used in communication.	4	Understanding
M1.03	Select various types of signals.	4	Apply
M1.04	Explain the basic operation on signals.	3	Understanding
Contents: • **Define signals:** state the importance of signals and systems in the field of science and engineering. • **Basic elementary signals:** Unit step function, unit impulse function, ramp, parabolic, signum, exponential, rectangular, triangular, and sinusoidal. • **Classification of signals:** continuous time and discrete time, deterministic and non-deterministic, even and odd, periodic and aperiodic, energy and power, real and imaginary. • **Mathematical operations on signals:** amplitude scaling, time scaling, time shifting, time reversal, addition, subtraction, multiplication and convolution.			
CO2	**Classify and compare continuous time and discrete time systems**		
M2.01	Show time domain representation of a system.	3	Understanding
M2.02	Compare continuous time and discrete time systems	3	Understanding
M2.03	Interpret impulse response of a continuous	3	Understanding

	time and discrete time system.		
M2.04	Identify various properties of systems.	3	Apply
	Series Test – I	1	

Contents:

- **Representation of systems:** Differential equation representation, Difference equation representation
- **Continuous time and discrete time systems –** Impulse response, examples
- **Properties of systems –** linearity, time invariant system, invertible, casual and non-casual, stable and unstable.

CO3	**Explain Fourier representation of signals**		
M3.01	Apply Fourier series and discrete time Fourier series	5	Apply
M3.02	Summarize the properties of Fourier series	3	Understanding
M3.03	Explain sampling theorem, aliasing, reconstruction	4	Understanding
M3.04	Apply Fourier transform, discrete time Fourier transform	4	Apply

Contents:

- Fourier representation of four class of signals
  - Continuous time periodic signal : Fourier series (FS)
  - Discrete time periodic signal : Discrete time Fourier series (DTFS)
  - Continuous time non-periodic signal : Fourier transform (FT)
  - Discrete time non-periodic signal : Discrete time Fourier transform (DTFT)
- Properties of Fourier representation – linearity, symmetry, time shift, frequency shift, scaling, differentiation and integration, convolution and modulation
- Sampling theorem, aliasing, reconstruction

CO4	**Apply Laplace transform to demonstrate the concepts of signals and systems**		
M4.01	Interpret the frequency domain parameters of a signal using Laplace transform	5	Understanding
M4.02	Illustrate the region of convergence	2	Understanding
M4.03	Outline Properties of Laplace transform	4	Understanding

M4.04	Apply Inverse Laplace transform to Signals	5	Applying
	Series Test – II	1	

Contents:

- **Need of Laplace transform**
- **Region of Convergence (ROC)**
- Advantages and limitation of Laplace transform
- **Laplace transform of some commonly used signals** - impulse, step, ramp, parabolic, exponential, sine and cosine functions
- **Properties of Laplace transform:** Linearity, time shifting, time scaling, time reversal, transform of derivatives and integrals, initial value theorem, final value theorem.
- **Inverse Laplace transform:** simple problems (no derivation required)

Text/Reference:

T/R	Book Title/Author
T1	Simon Haykin : Signals and System, John Wiley 2/e 2003
T2	A. Anand Kumar : Signals and System, PHI, 2/e 2012
T3	Nagoor Kani : Signals and System, Tata McGrew Hill, 3/e 2011
R1	Ramesh Babu : Signals and Systems, Scotch Publications, 4/e
R2	Alan V. Oppenhein, Alan S. Willsky : Signals and System, PHI, 2/e
R3	Simon Haykin : Communication Systems, John Wiley 4/e
R4	Hwei P. Hsue : Signals and System, Tata McGrew Hill 1995
R5	Rodger E. Ziemer : Signals and System – Continuous and Discrete 4/e, Pearson Education
R6	M. J. Robert : Signals and Systems, Tata McGrew Hill, 2003

Online Resources:

Sl.No	Website Link
1	https://nptel.ac.in/courses/108/104/108104100/
2	https://nptel.ac.in/courses/117/101/117101055/

Revision 2021

SIGNALS & SYSTEMS

MODULE 1 NOTES

OUTCOMES & CONTENTS

Module Outcomes	**Description**	**Duration (Hours)**	**Cognitive Level**
CO1	**Summarize the basic concepts, classifications and mathematical propertiesof signals**		
M1.01	Define signals and its importance.	3	Understanding
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M1.04	Explain the basic operation on signals	3	Understanding

Contents:

Define signals: state the importance of signals and systems in the field of science and engineering.

Basic elementary signals: Unit step function, unit impulse function, ramp, parabolic, signum, exponential, rectangular, triangular, and sinusoidal.

Classification of signals: continuous time and discrete time, deterministic and nondeterministic, even and odd, periodic and aperiodic, energy and power, real and imaginary.

Mathematical operations on signals: amplitude scaling, time scaling, time shifting, time reversal, addition, subtraction, multiplication and convolution.

SIGNALS

- A function of one or more independent variables which contains some information is called a signal.
- Eg.: speech signal, E.C.G. signals, temperature variations, etc.
- A voltage or current is an example of a signal which is a function of time as an independent variable.
- An AC signal is generally represented in the form: -

$$x(t) = Acos(\omega t + \phi)$$

  x(t) = independent variables
  A = amplitude
  ω = angular frequency in radians
  ϕ = phase angle in radians

Depending on the number of variables signals can be:

- 1-dimensional signal: function of a single variable.
  Eg: speech signal represented by f(t).
- 2-dimensional signal: function of two variables.
  Eg: image signal represented by f(x,y)
- 3-dimensional signal: function of three variables.
  Eg: video signal represented by f(x,y,t)

BASIC ELEMENTARY SIGNALS

They are the building blocks for the construction of more complex signals. Also called *standard signals.*

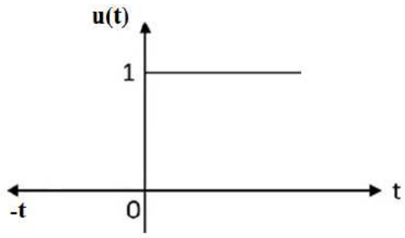
1. Unit Step Function
2. Unit Impulse Function
3. Unit Ramp Function
4. Parabolic Function
5. Signum Function
6. Exponential Signal
7. Rectangular Function
8. Triangular Function
9. Sinusoidal Signal

1. **Unit Step Function**

Exists only for positive values of time and is zero for negative time. If a step function has unit magnitude it is called unit step function.

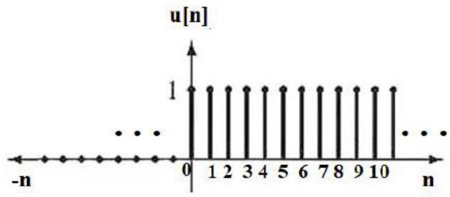
Continuous time

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



Discrete time

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

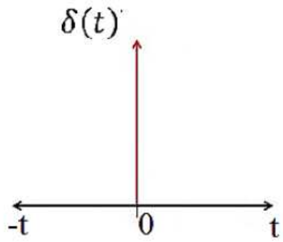


2. **Unit Impulse Function**

Continuous time

Continuous time unit impulse function is also called Dirac delta function.
It is only defined at t=0.

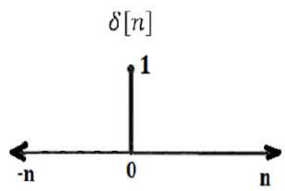
$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



Discrete time

Discrete time unit impulse function is also called unit sample sequence.
It is defined only at n=0.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



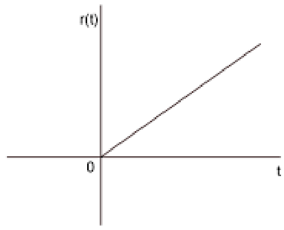
3. **Unit Ramp Function**

Amplitude increases linearly with time. Unit ramp function has unit slope.

Continuous time

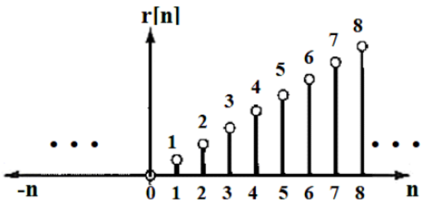
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$r(t) = t.u(t)$$



Discrete time

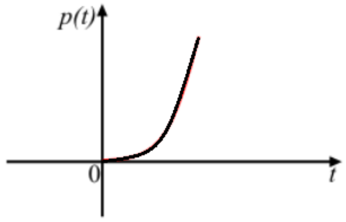
$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



4. **Parabolic Function**

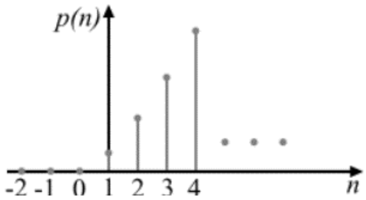
Continuous time

$$p(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Discrete time

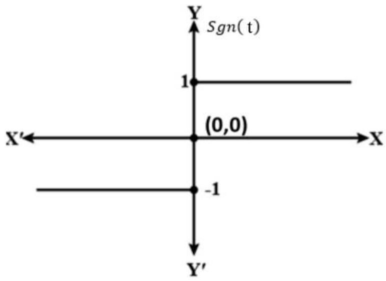
$$p[n] = \begin{cases} \frac{n^2}{2}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



5. **Signum Function**

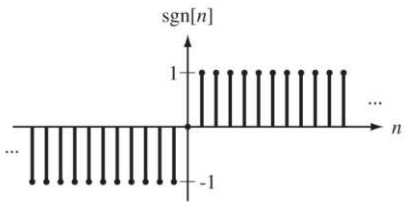
Continuous time

$$Sgn(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$



Discrete time

$$\mathrm{sgn}[n] = \begin{cases} 1, & n > 0 \\ 0, & n = 0 \\ -1, & n < 0 \end{cases}$$



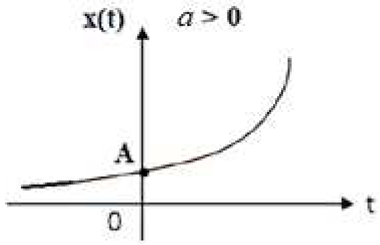
6. **Exponential Signal**

Continuous time

$$x(t) = Ae^{at}$$

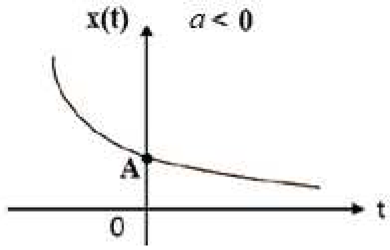
'A' and 'a' are real numbers. 'A' is the amplitude of the exponential signal measured at t= 0.
'a' can be either positive or negative.
At t = -α, $x(t) = Ae^{at}$ becomes $x(t) = Ae^{a.-\alpha}$ or $x(t) = Ae^{-\alpha}$ = 0
At t = 0, $x(t) = Ae^{at}$ becomes $x(t) = Ae^{a.0}$ or $x(t) = Ae^{0}$ = 1
At t = α, $x(t) = Ae^{at}$ becomes $x(t) = Ae^{a.\alpha}$ or $x(t) = Ae^{\alpha}$ = α

A. Rising when 'a' is positive



$$x(t) = \begin{cases} 0\ at\ t = -\alpha: Ae^{-\alpha} = 0 \\ 1\ at\ t = 0 : Ae^{0} = 1 \\ \alpha\ at\ t = \alpha: Ae^{\alpha} = \alpha \end{cases}$$

B. Decaying when 'a' is negative



$$x(t) = \begin{cases} \alpha\ at\ t = -\alpha: Ae^{\alpha} = \alpha \\ 1\ at\ t = 0 : Ae^{0} = 1 \\ 0\ at\ t = \alpha: Ae^{-\alpha} = 0 \end{cases}$$

Discrete time

$$x[n] = Ae^{an}$$

'A' and 'n' are real numbers. 'A' is the amplitude of the exponential signal measured at t= 0.
'n' can be either positive or negative.
At n = -α, $x[n] = Ae^{an}$ becomes $x[n] = Ae^{a.-\alpha}$ or $x[n] = Ae^{-\alpha}$ = 0
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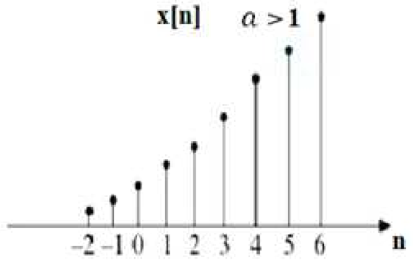
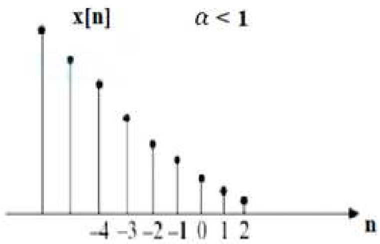
At n = α, $x[n] = Ae^{an}$ becomes $x[n] = Ae^{a.\alpha}$ or $x[n] = Ae^{\alpha}$ = α

A. Rising when ‘a’ is positive



$$x[n] = \begin{cases} 0\ at\ n = -\alpha: Ae^{-\alpha} = 0 \\ 1\ at\ n = 0: Ae^{0} = 1 \\ \alpha\ at\ n = \alpha: Ae^{\alpha} = \alpha \end{cases}$$

B. Decaying when ‘a’ is negative

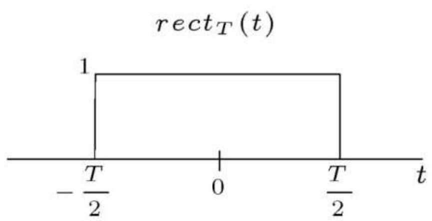


$$x[n] = \begin{cases} \alpha\ at\ n = -\alpha: Ae^{-\alpha} = \alpha \\ 1\ at\ n = 0: Ae^{0} = 1 \\ 0\ at\ n = \alpha: Ae^{-\alpha} = 0 \end{cases}$$

7. Rectangular Function

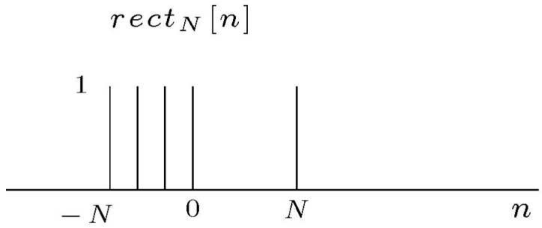
Continuous time

$$rect(t) = \begin{cases} 1 & \text{if } \frac{-1}{2} \geq t \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Discrete time

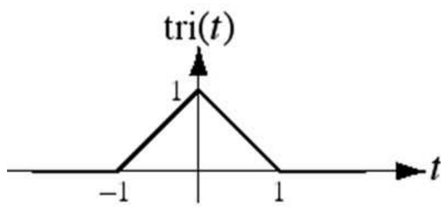
$$rect[n] = \begin{cases} 1 & \text{if } \frac{-1}{2} \geq n \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



8. Triangular Function

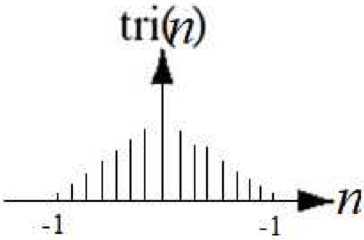
Continuous time

$$\text{tri}(t) = \begin{cases} 1 - |t| & , |t| < 1 \\ 0 & , |t| \geq 1 \end{cases}$$



Discrete time

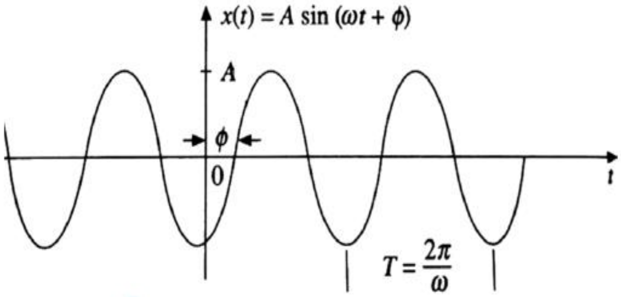
$$\text{tri}[n] = \begin{cases} 1-|n| & , \ |n|<1 \\ 0 & , \ |n|\geq 1 \end{cases}$$



9. **Sinusoidal Signal**

Continuous time

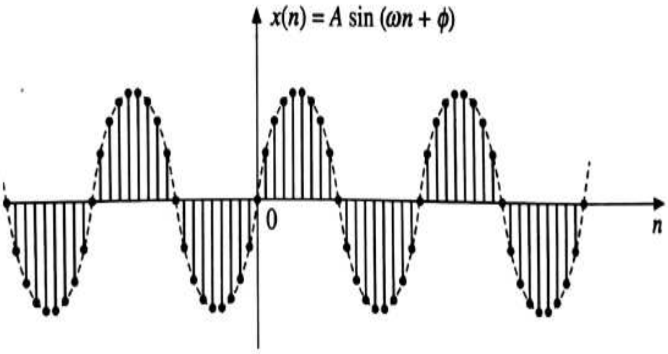
x(t)=A sin (ωt+ϕ)



x(t) = independent variables
A = amplitude
ω = angular frequency in radians
T = $\frac{2\pi}{\omega}$
ϕ = phase angle in radians

Discrete time

X[n] = A sin (ωn+ϕ)

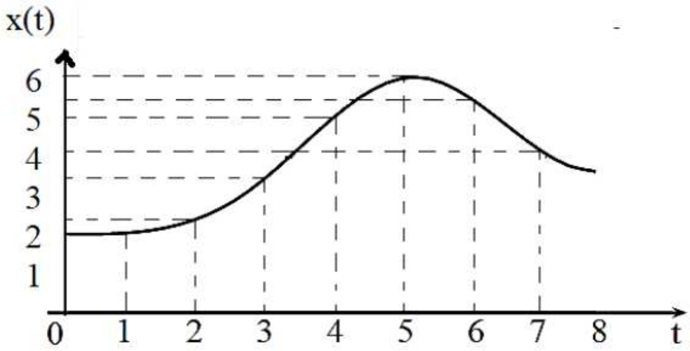


X[n] = independent variables
A = amplitude
ω = angular frequency in radians
N = $\frac{2\pi}{\omega}.K$
ϕ = phase angle in radians

CLASSIFICATION OF SIGNALS

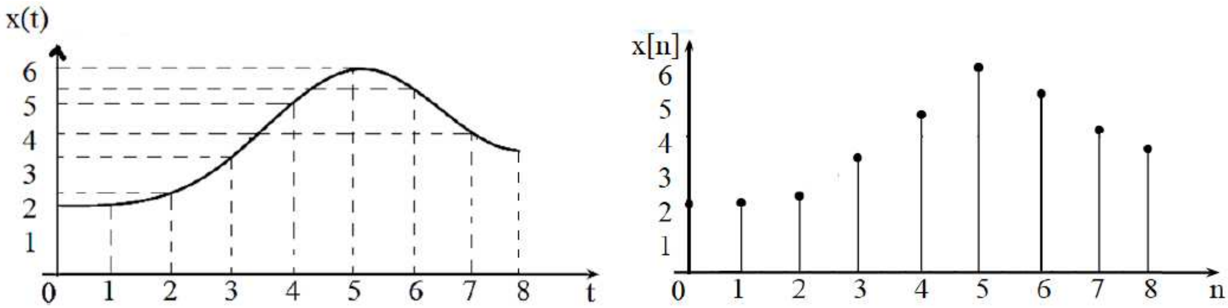
1. Continuous time signals:

- These signals are defined over continuous independent variables.
- They are continuous in time and amplitude.
- Generally denoted by x(t).
- They are difficult to analyze, as they carry a huge number of values. In order to store these signals, we require an infinite memory.



2. Discrete time signals:

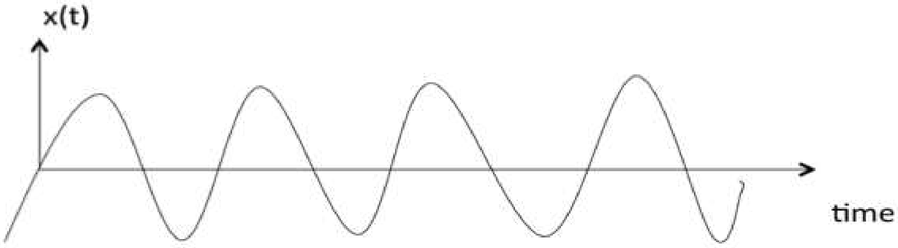
- These signals are defined for only discrete values of time denoted by n.
- Discrete time signals are defined for only integer values of n.
- Obtained by sampling continuous time signals.
- $x[n] = x(t)|_{t=nT}$, where T is the sampling time.
- They are continuous in amplitude and discrete in time and is denoted by x[n]



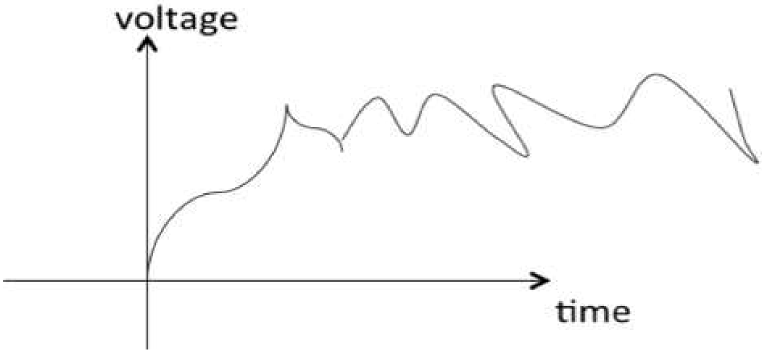
In the above example, the sampling time interval was T=1sec, which means samples are taken after every 1 sec to form the discrete time signal.

3. Deterministic and non-deterministic

- A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.

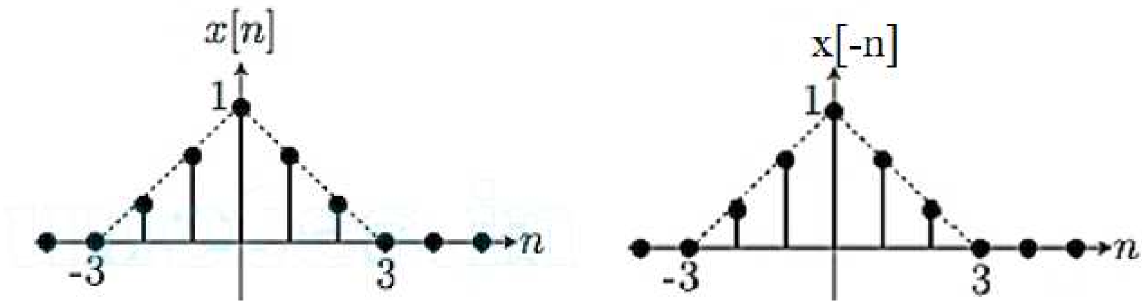


- A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.

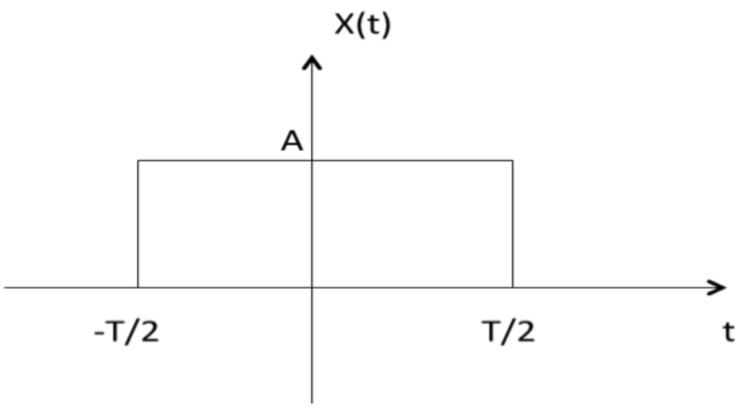


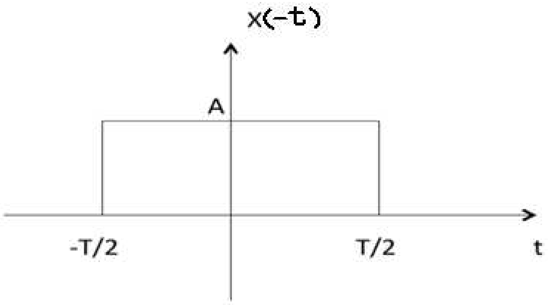
4. Even and odd

A signal is said to be even when it satisfies the condition **x(t) = x(-t) or x[n] = x[-n]**

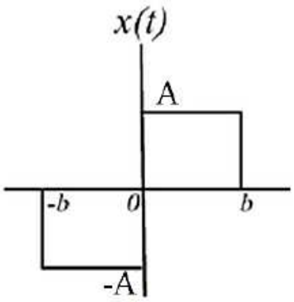


Example: As shown in the following diagram, rectangle function x(t) = x(-t) so it is even function.

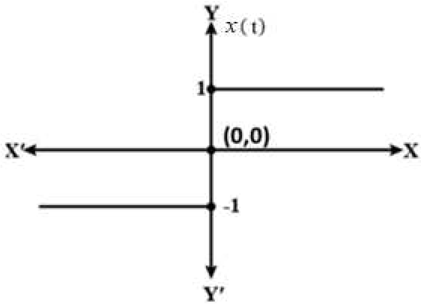


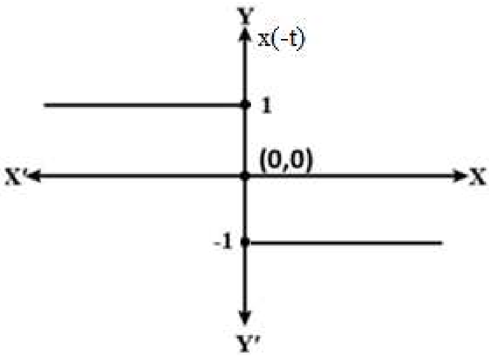


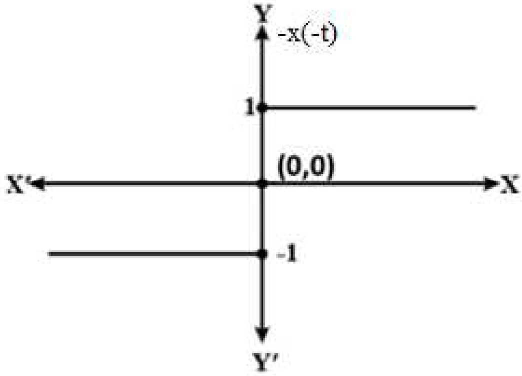
❖ A signal is said to be odd when it satisfies the condition **x(t) = -x(-t) or x[n] = - x[-n]**



Example: As shown in the following diagram, signum function x(t) = - x(-t) so it is odd function.







Any function ƒ(t) can be expressed as the sum of its even function $f_e(t)$ and odd function $f_o(t)$.

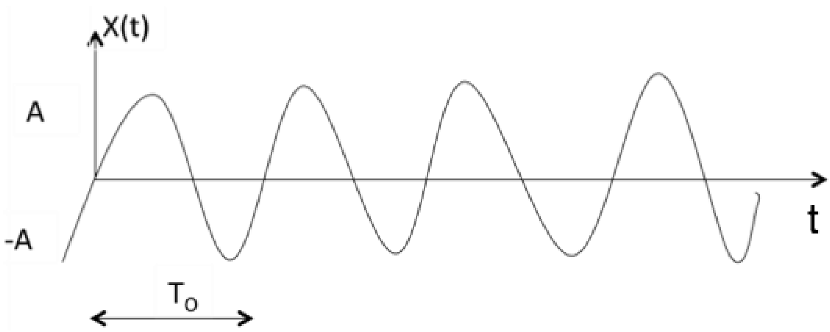
$$f(t) = f_e(t) + f_o(t)$$

where

$$f_e(t) = ½[f(t) + f(-t)] \text{ and } f_o(t) = ½[f(t) - f(-t)]$$

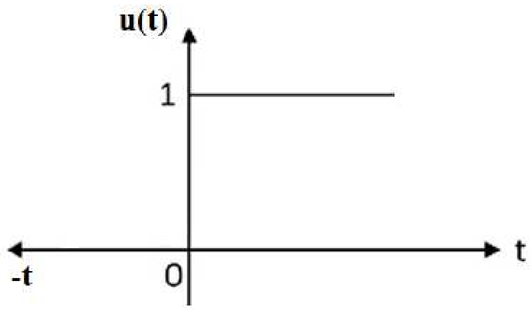
5. **Periodic and aperiodic**
   - ❖ A signal is said to be periodic signal if it has a definite pattern and repeats itself at a regular interval of time.
   - ❖ A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N).

   where
   T = fundamental time period,
   1/T = f = fundamental frequency.



The above signal will repeat for every time interval T_0 hence it is periodic with period T_0.

❖ A signal is said to be aperiodic signal if it does not have a definite pattern and does not repeat itself at a regular interval of time.



6. **Energy and power**

- A signal is said to be energy signal when it has finite energy.

$$E_{x(t)} = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \,dt$$

$$0 < E_x(t) < \alpha$$

$$E_{x(n)} = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$0 < E_x(n) < \alpha$$

- A signal is said to be power signal when it has finite power.

$$P_{av}x(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \,dt$$

$$0 < P_{av}x(t) < \alpha$$

$$P_{av}x(n) = \lim_{N\to\infty} \frac{1}{2n+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$0 < P_{av}x(n) < \alpha$$

7. **Real and imaginary**

- A signal is said to be real when it satisfies the condition x*(t) = x(t)

  Example:

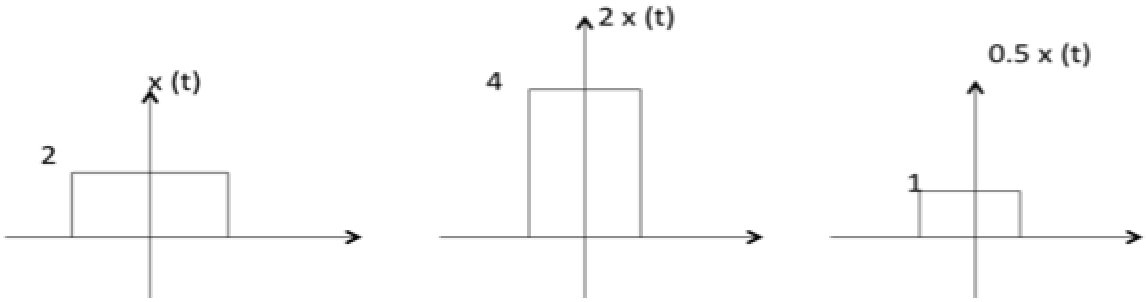
  If x(t)= 3 then x*(t)=3*=3 here x(t) is a real signal.

- A signal is said to be imaginary when it does not satisfy the condition x*(t) = x(t)
  - For a real signal, imaginary part should be zero.
  - Similarly for an imaginary signal, real part should be zero.

MATHEMATICAL OPERATIONS ON SIGNALS

Amplitude scaling

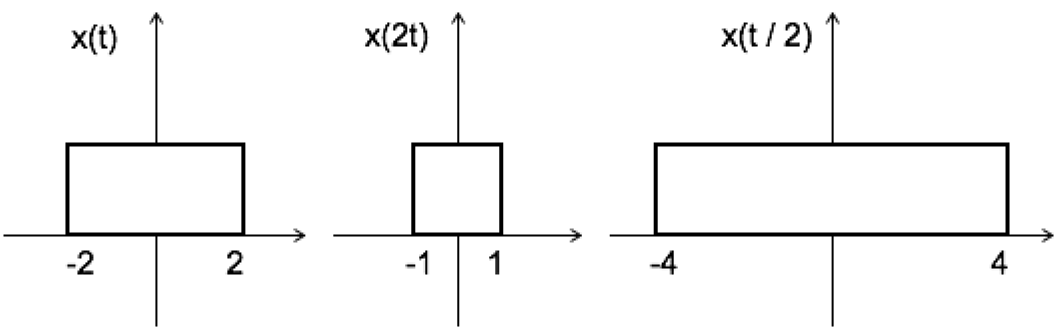
Cx(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



Time scaling

x(At) is time scaled version of the signal x(t). where A is always positive.

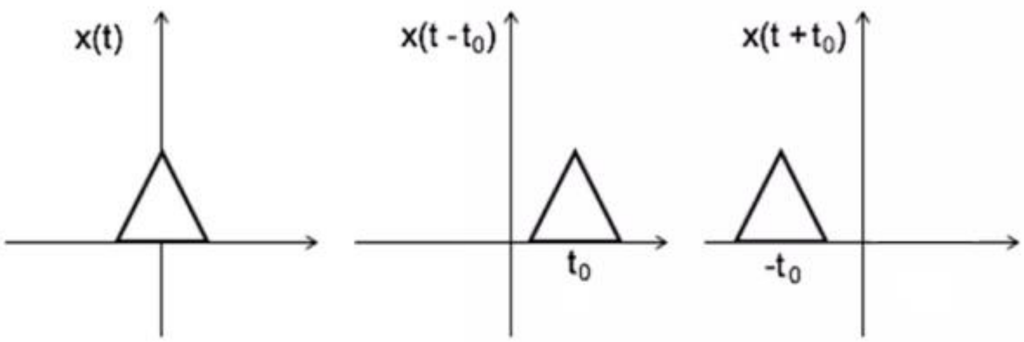
|A| > 1 → Compression of the signal
|A| < 1 → Expansion of the signal



Note: u(at) = u(t) time scaling is not applicable for unit step function.

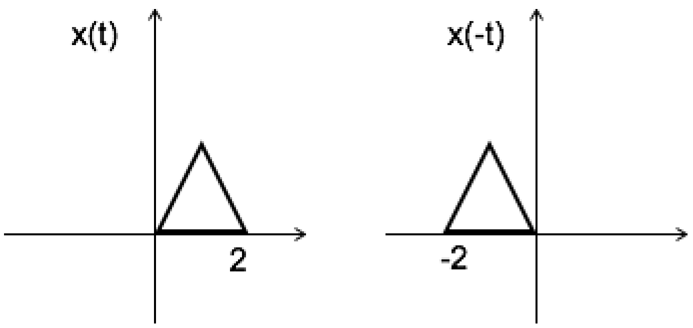
Time shifting

$x(t \pm t_0)$ is time shifted version of the signal x(t).
$x(t + t_0)$ → negative shift
$x(t - t_0)$ → positive shift



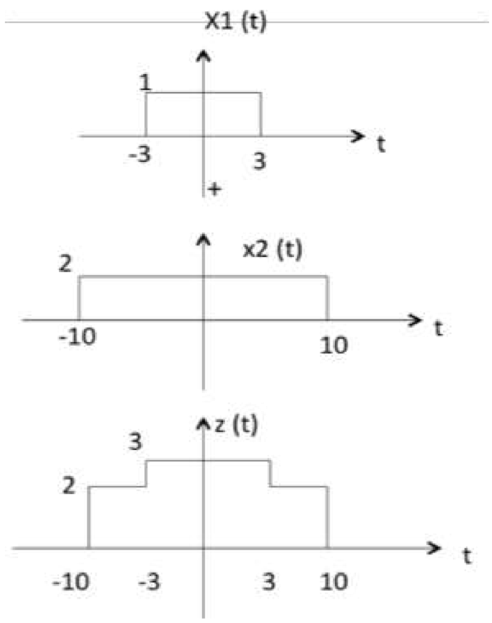
Time reversal

x(-t) is the time reversal of the signal x(t).



Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



As seen from the diagram above,

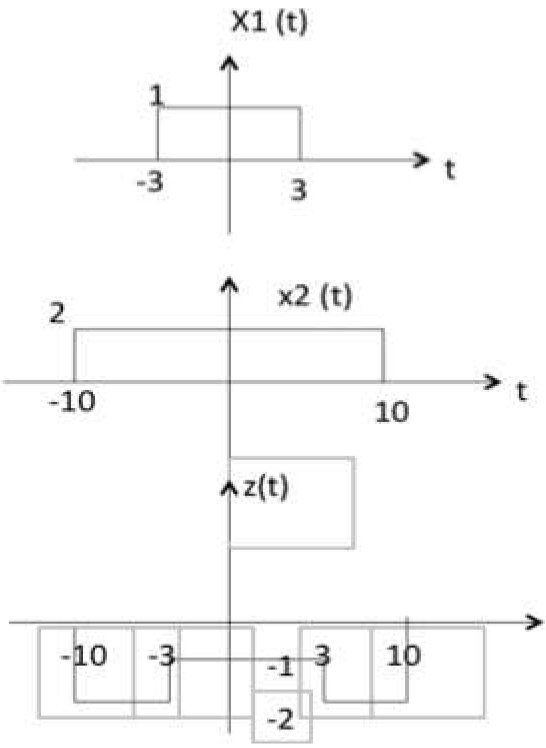
$-10 < t < -3$ amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

$-3 < t < 3$ amplitude of z(t) = x1(t) + x2(t) = 1 + 2 = 3

$3 < t < 10$ amplitude of z(t) = x1(t) + x2(t) = 0 + 2 = 2

Subtraction

Subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

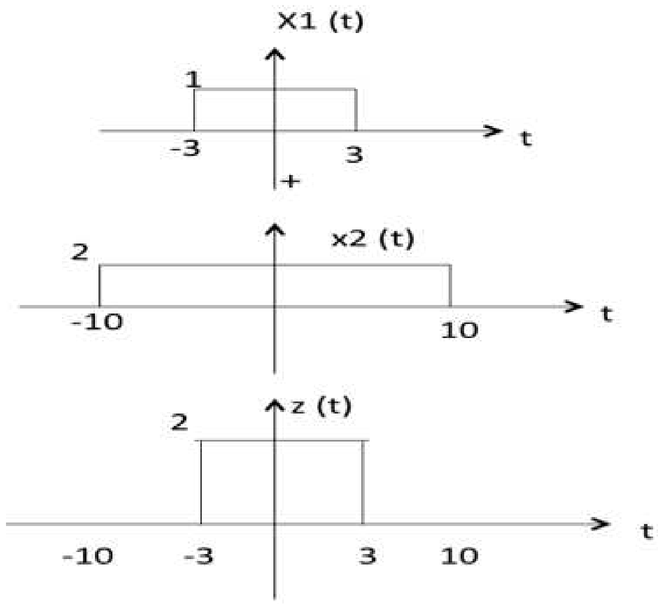
-10 < t < -3 amplitude of z (t) = x1(t) - x2(t) = 0 - 2 = -2

-3 < t < 3 amplitude of z (t) = x1(t) - x2(t) = 1 - 2 = -1

3 < t < 10 amplitude of z (t) = x1(t) + x2(t) = 0 - 2 = -2

Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

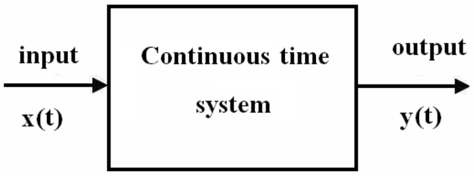
$-10 < t < -3$ amplitude of z (t) = $x_1(t) \times x_2(t) = 0 \times 2 = 0$

$-3 < t < 3$ amplitude of z (t) = $x_1(t) \times x_2(t) = 1 \times 2 = 2$

$3 < t < 10$ amplitude of z (t) = $x_1(t) \times x_2(t) = 0 \times 2 = 0$

Convolution

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).



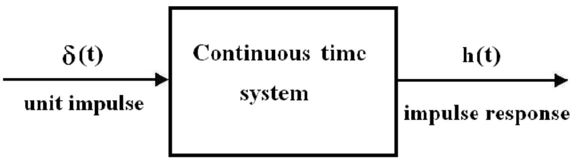
Some of the different methods of representing the continuous time system are:

i. Differential equation
ii. Block diagram
iii. Impulse response
iv. Frequency response
v. Laplace-transform
vi. Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

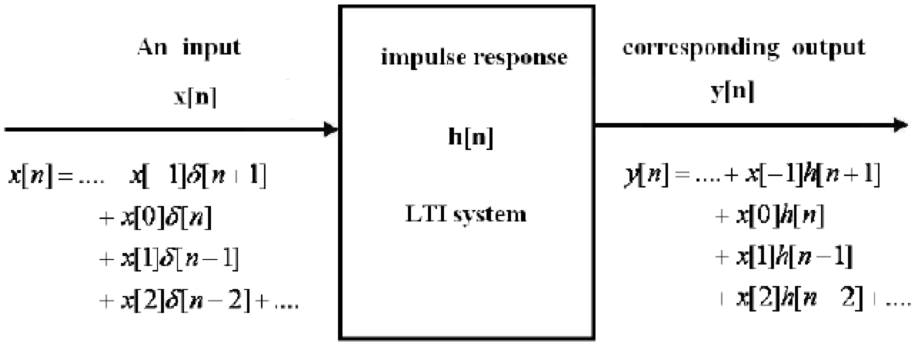
<u>Impulse Response</u>

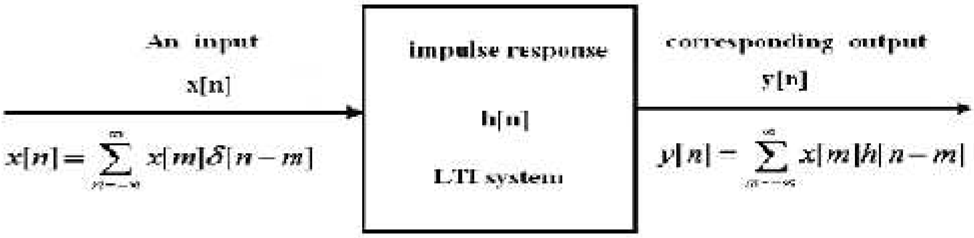
The impulse response of a continuous time system is defined as the output of the system when its input is a unit impulse, δ(t). Usually, the impulse response is denoted by h(t).



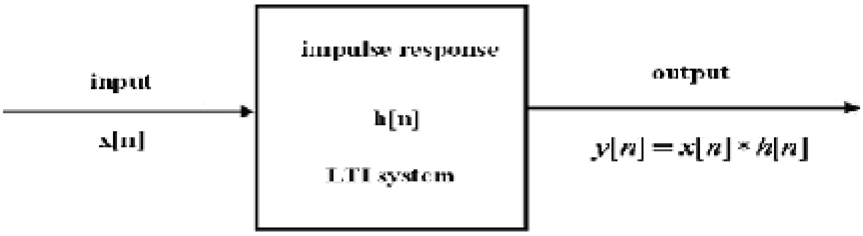
Convolution Sum:

We now attempt to obtain the output of a digital system for an arbitrary input x[n], from the knowledge of the system impulse response h[n].





time-domain analysis



$$y[n] = x[n] * h[n]$$

Evaluating the convolution sum:

Given the system impulse response h[n], and the input x[n], the system output y[n], is given by the convolution sum:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

<u>Problem:</u>

To obtain the digital system output y[n], given the system impulse response h[n], and the system input x[n] as:

h[n]=[1, -1.5, 3]

$$x[n] = [-1, \underset{\uparrow}{2.5}, 0.8, 1.25]$$

-1 4 -5.95 7.55 0.525 3.75

Evaluation using graphical representation:

This method is based on evaluation of the convolution sum for a single value of n, and varying n over all possible values.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Step 1: Sketch x[m]

Step 2: Sketch h[-m]

Step 3: Compute y[0] using:

$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[-m]$$

which is the 'sum of the product of the two signals x[m] & h[-m]'

Step 4: Sketch h[1-m], which is right shift of h[-m] by 1.

Step 5: Compute y[1] using:

$$y[1] = \sum_{m=-\infty}^{\infty} x[m]h[1-m]$$

which is the 'sum of the product of the two signals x[m] & h[1-m]'

Step 6: Sketch h[2-m], which is right shift of h[-m] by 2.

Step 7: Compute y[2] using:

$$y[2] = \sum_{m=-\infty}^{\infty} x[m]h[2-m]$$

which is the 'sum of the product of the two signals x[m] & h[2-m]'

Step 8: Proceed this way until all possible values of y[n], for positive 'n' are computed

Step 9: Sketch h[-1-m], which is left shift of h[-m] by 1.

Step 10: Compute y[-1] using:

$$y[-1] = \sum_{m=-\infty}^{\infty} x[m]h[-1-m]$$

which is the 'sum of the product of the two signals x[m] & h[-1-m]'

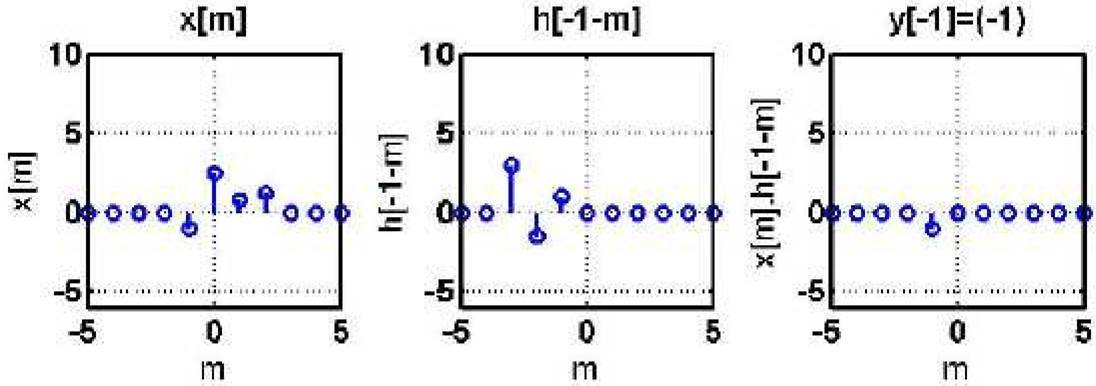
Step 11: Sketch h[-2-m], which is left shift of h[-m] by 2.

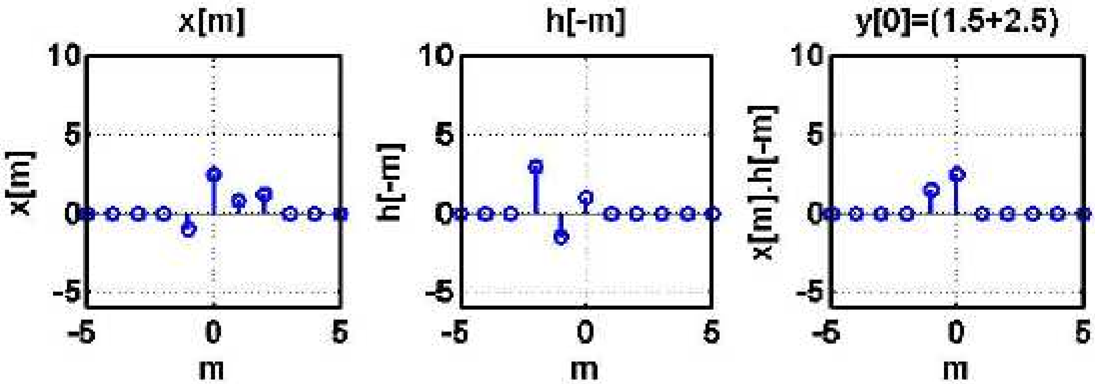
Step 12: Compute y[-2] using:

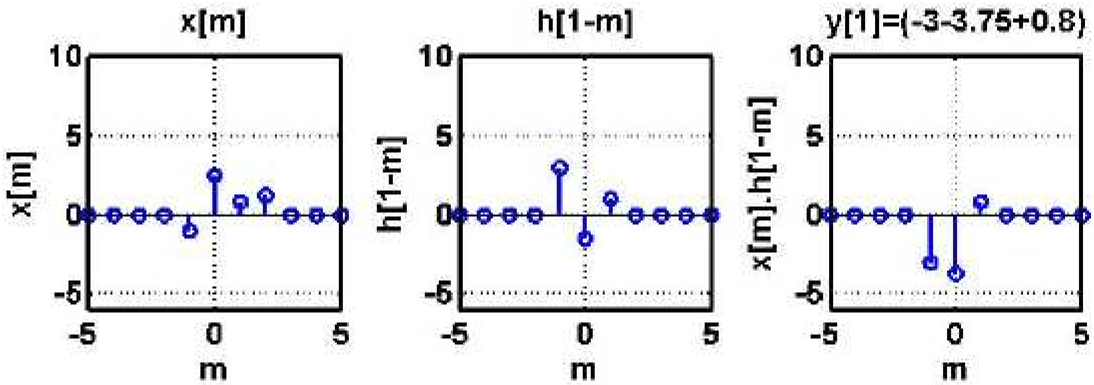
$$y[-2] = \sum_{m=-\infty}^{\infty} x[m]h[-2-m]$$

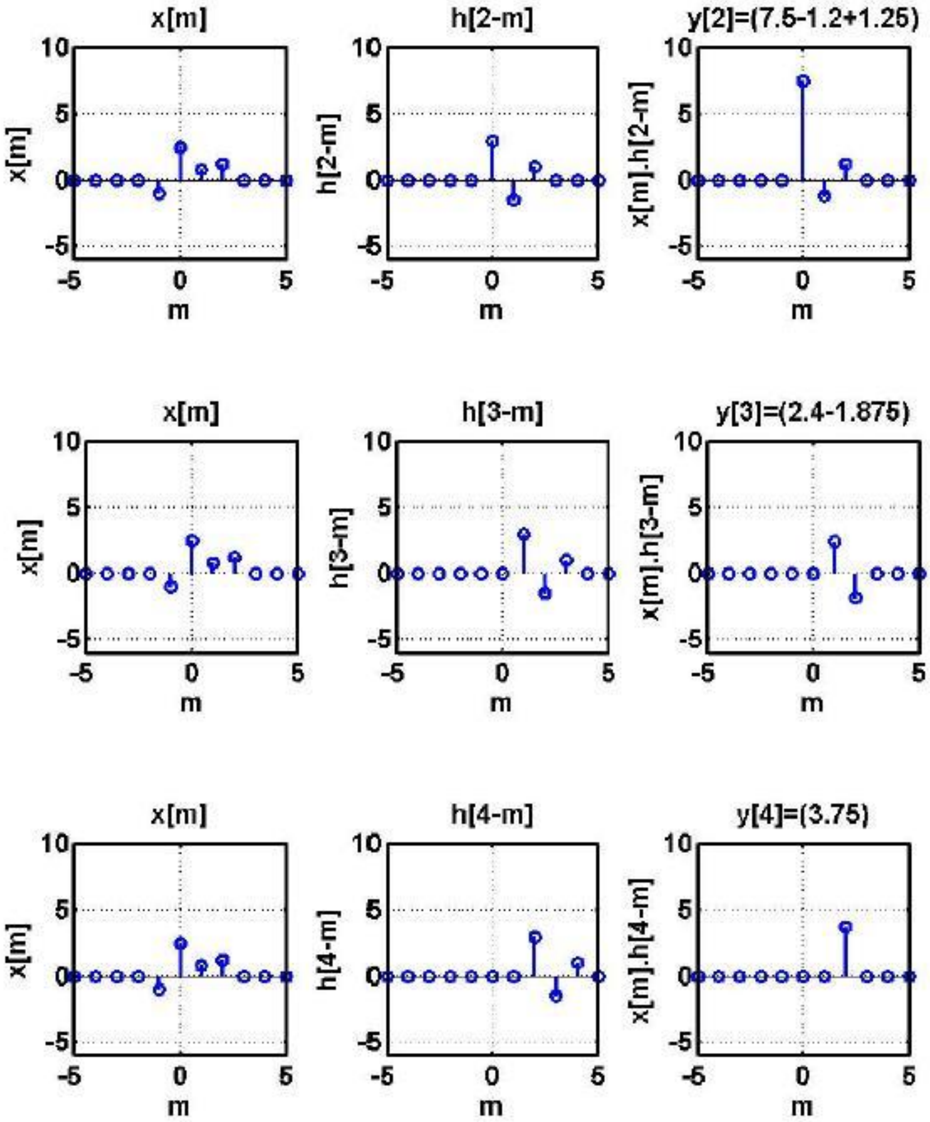
which is the 'sum of the product of the two signals x[m] & h[-2-m]'

Step 13: Proceed this way until all possible values of y[n], for negative 'n' are computed

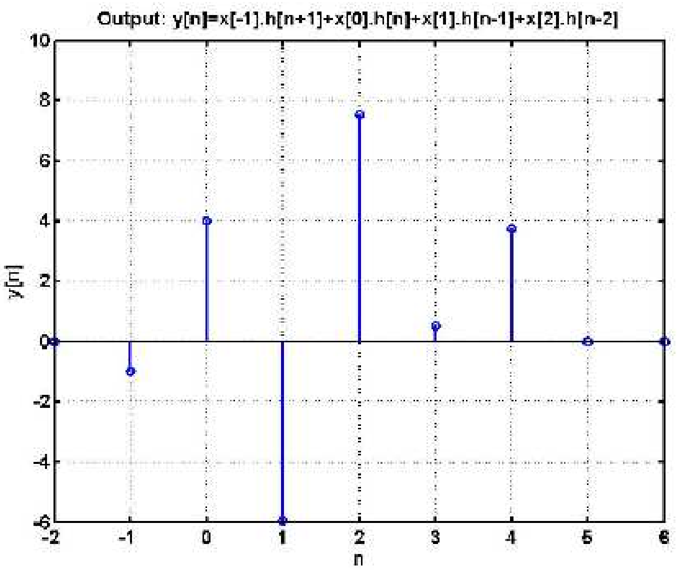

x[m]
h[-1-m]
y[-1]=(-1)
x[m]
h[-1-m]
x[m].h[-1-m]
m


x[m]
h[-m]
y[0]=(1.5+2.5)
x[m]
h[-m]
x[m].h[-m]
m


x[m]
h[1-m]
y[1]=(-3-3.75+0.8)
x[m]
h[1-m]
x[m].h[1-m]
m



x[m]
h[2-m]
y[2]=(7.5-1.2+1.25)
h[2-m]
x[m].h[2-m]
x[m]
h[3-m]
y[3]=(2.4-1.875)
h[3-m]
x[m].h[3-m]
x[m]
h[4-m]
y[4]=(3.75)
h[4-m]
x[m].h[4-m]
m

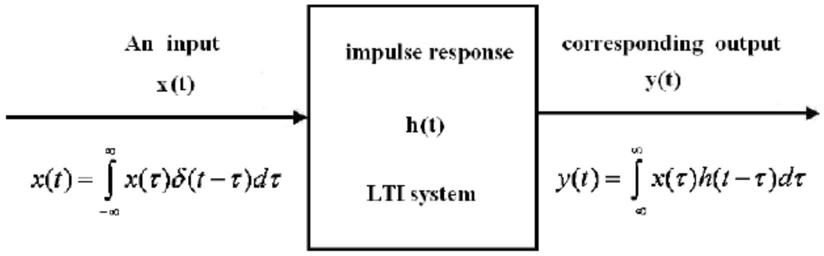


Output: y[n]=x[-1].h[n+1]+x[0].h[n]+x[1].h[n-1]+x[2].h[n-2]
y[n]
n

Convolution Integral

The output y(t) is given by, using the notation, y(t)=R{x(t)}.

$$y(t) = R\{x(t)\}$$
$$= R\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$
$$= \int_{-\infty}^{\infty} x(\tau)R\{\delta(t-\tau)\}d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$= x(t) * h(t)$$



Example 1:

Consider the convolution of the delta impulse signal and any other regular signal f(t).

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau, \quad -\infty < t < \infty$$

$$f(t) * \delta(t) = f(t)$$

Example 2:

Consider the convolution of e^{-t} u(t) and sin(t)

$$e^{-t}u(t) * \sin(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\sin(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)\sin(t-\tau)d\tau$$

$$e^{-t}u(t) * \sin(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}\sin(\tau)d\tau$$

$$= e^{-t}\left[\frac{e^{t}}{2}(\sin(t) - \cos(t)) - 0\right] = \frac{1}{2}(\sin(t) - \cos(t))$$

Graphical convolution

The graphical presentation of the convolution integral helps in the understanding of every step in the convolution procedure. According to the definition integral, the convolution procedure involves the following steps:

Step 1: Apply the convolution duration property to identify intervals in which the convolution is equal to zero.

Step 2: Flip about the vertical axis one of the signals (the one that has a simpler form (shape) since the commutativity holds), that is, represent one of the signals in the time scale $-\tau$.

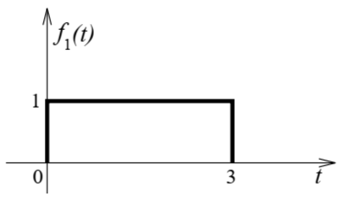
Step 3: Vary the parameter t from -infinity to +infinity, that is, slide the flipped signal from the left to the right, look for the intervals where it overlaps with the other signal, and evaluate the integral of the product of two signals in the corresponding intervals.

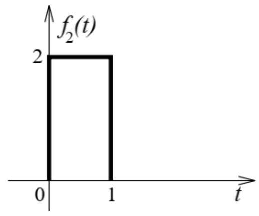
In the above steps one can also incorporate (if applicable) the convolution time shifting property such that all signals start at the origin. In such a case, after the final convolution result is obtained the convolution time shifting formula should be applied appropriately. In addition, the convolution continuity property may be used to check the obtained convolution result, which requires that at the boundaries of adjacent intervals the convolution remains a continuous function of the parameter.

We present several graphical convolution problems starting with the simplest one.

Example:

Consider two rectangular pulses given below





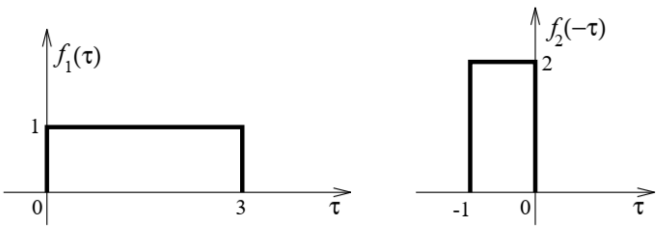
Since the durations of the signals $f_1(t)$ and $f_2(t)$ are respectively given by $[t_1, T_1] = [0,3]$ and $[t_2, T_2] = [0,1]$, we conclude that the convolution of these two signals is zero in the following intervals (Step 1).

$$f_1(t) * f_2(t) = 0, \quad t \leq t_1 + t_2 = 0 + 0 = 0$$

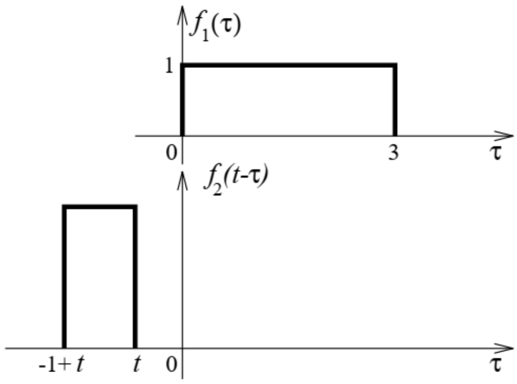
$$f_1(t) * f_2(t) = 0, \quad t \geq T_1 + T_2 = 1 + 3 = 4$$

Thus, we need only to evaluate the convolution integral in the interval $0 \leq t \leq 4$.

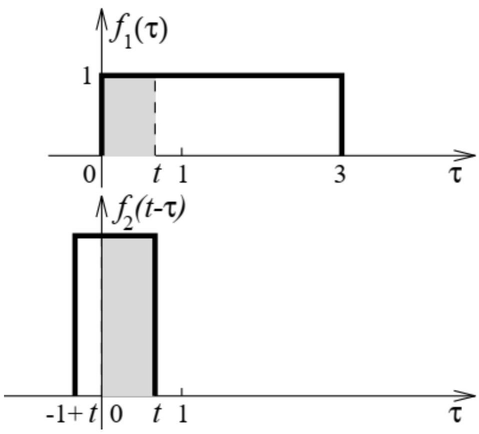
In the second step, we flip about the vertical axis the signal which has a simpler shape. Since in this case both signals are rectangular pulses, it is irrelevant which one is flipped. Let us flip $f_2(t)$. Note that the convolution is performed in the time scale.



In Step 3, we shift the signal $f_2(-\tau)$ to the left and to the right, that is, we form the signal $f_2(t-\tau)$ for $t \in (-\infty, 0]$ and $t \in [0, +\infty)$. A shift of the signal $f_2(t-\tau)$ to the left (t<0) produces no overlapping between the signals $f_1(\tau)$ and $f_2(t-\tau)$, thus the convolution integral is equal to zero.



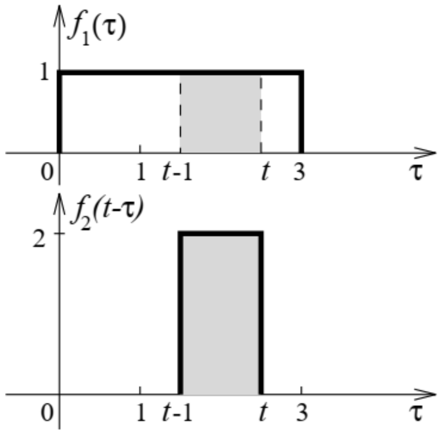
Let us start shifting the signal $f_2(t-\tau)$ to the right (t>0). Consider first the interval 0 ≤ t ≤ 1.



It can be seen from Figure that in the interval from zero to the signals overlap, hence their product is different from zero in this interval, which implies that the convolution integral is given by

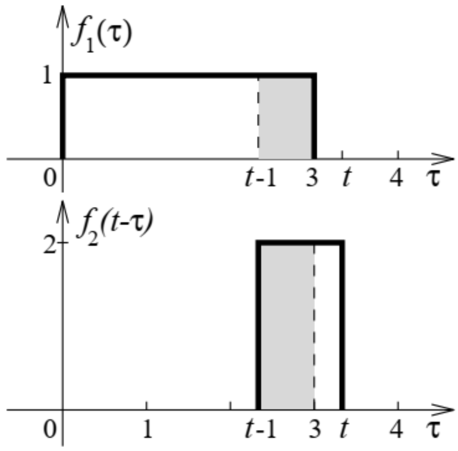
$$f_1(t) * f_2(t) = \int_0^t 1 \times 2d\tau = 2t, \quad 0 \le t \le 1$$

By shifting the signal $f_2(t-\tau)$ further to the right, we get the same “kind of overlap” for 1 ≤ t ≤ 3,



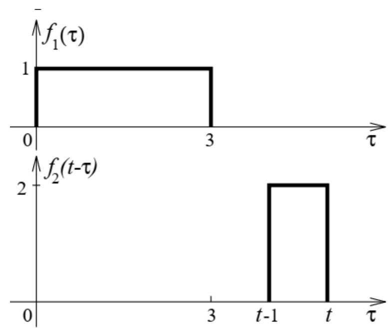
$$f_1(t) * f_2(t) = \int_{t-1}^{t} 1 \times 2 d\tau = 2, \quad 1 \leq t \leq 3$$

By shifting $f_2(t-\tau)$ further to the right, for 3 ≤ t ≤ 4, we get the situation presented below.



$$f_1(t) * f_2(t) = \int_{t-1}^{3} 1 \times 2 d\tau = 8 - 2t, \quad 3 \leq t \leq 4$$

For t<4, the convolution is equal to zero as determined in Step 1. This can be justified by the fact that the signals $f_1(\tau)$ and $f_2(t-\tau)$ do not overlap for that is, their product is equal to zero for t>4, which implies that the corresponding integral is equal to zero in the same interval.

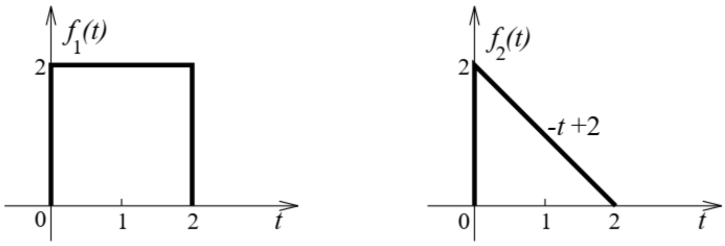


In summary, the convolution of the considered signals is given by

$$f_1(t) * f_2(t) = \begin{cases} 0 & t \leq 0 \\ 2t & 0 \leq t \leq 1 \\ 2 & 1 \leq t \leq 3 \\ 8 - 2t & 3 \leq t \leq 4 \\ 0 & t \geq 4 \end{cases}$$

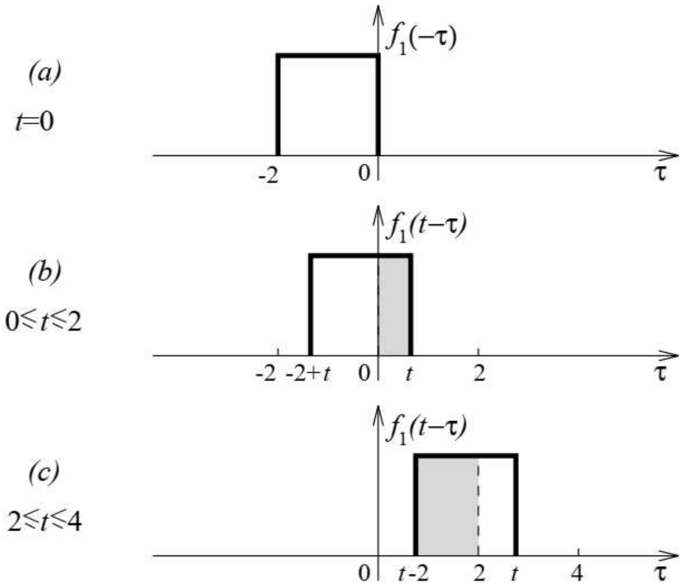
Example 2

Let us convolve the signals



Since both signals have the duration intervals from zero to two, we conclude that the convolution integral is zero for t≤0 and t≥4.

In the next step we flip about the vertical axis the rectangular signal since it apparently has a simpler shape. In Step 3, we slide the rectangular signal to the right for $t \in [0,2]$, Figure b, and for $t \in [2,4]$, Figure c.



The convolution integral in these two intervals, evaluated according to information given in Figures b and c, is respectively given by

$$f_1(t) * f_2(t) = \int_0^t 2(-\tau + 2)d\tau = 4t - t^2, \quad 0 \le t \le 2$$

$$f_1(t) * f_2(t) = \int_{t-2}^{2} 2(-\tau + 2)d\tau = 16 - 8t + t^2, \quad 2 \le t \le 4$$

In summary, we have obtained

$$f_1(t) * f_2(t) = \begin{cases} 0 & t \le 0 \\ 4t - t^2 & 0 \le t \le 2 \\ 16 - 8t + t^2 & 2 \le t \le 4 \\ 0 & t \ge 4 \end{cases}$$

SOLVED PROBLEMS

1. **What is a gate function?**

   Rectangular function $A\ rect\left(\frac{t}{T}\right)$ is called a gate function.

2. **What is a dirac delta function?**

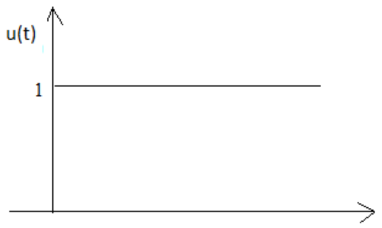
   Impulse function $\delta(t)$ is called a dirac delta function.

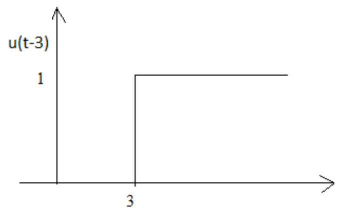
3. **If u(t) is unit step function, draw**

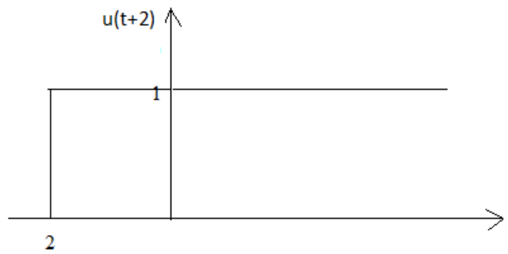
   a) u(t-3)

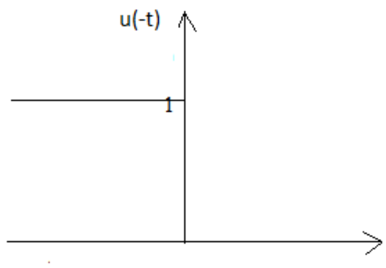
   b) u(t+2)

   c) u(-t)

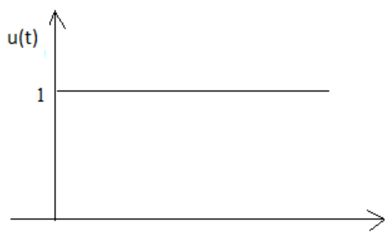


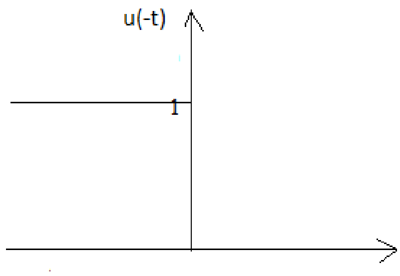


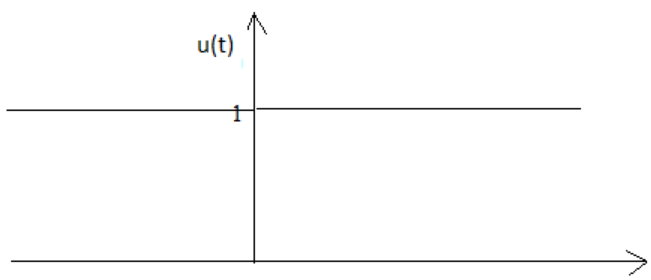




4. **What is the value of u(t) + u(-t)?**

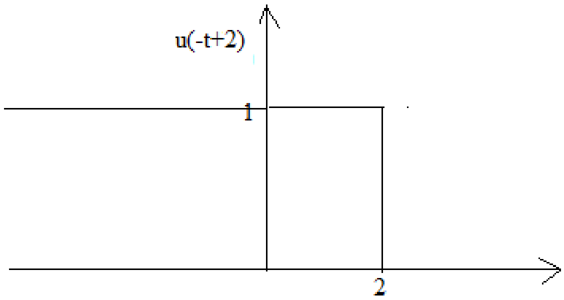






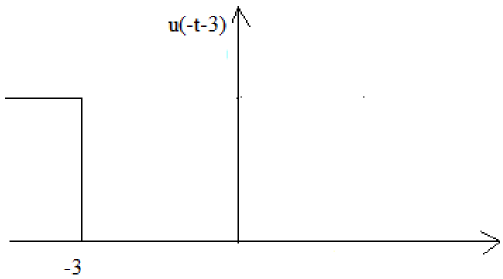
Therefore u(t) + u(-t) = 1

5. **Draw u(-t+2) using reversal and shifting operations.**

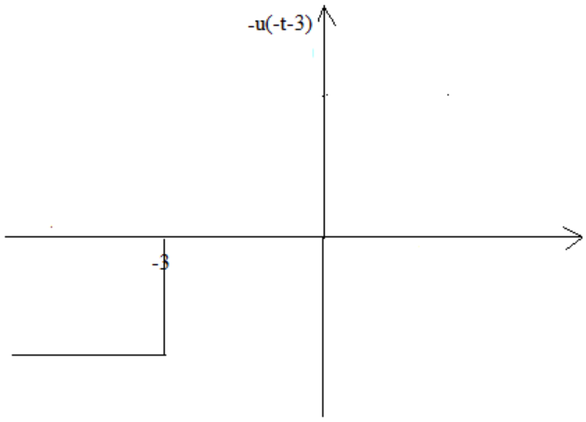


(always do reversal first, then shifting) ie u(-(t-2))

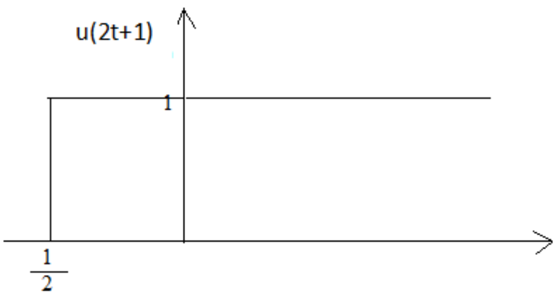
6. **Draw u(-t-3) using reversal and shifting operations.**



7. **Draw - u(-t-3) using reversal and shifting operations.**

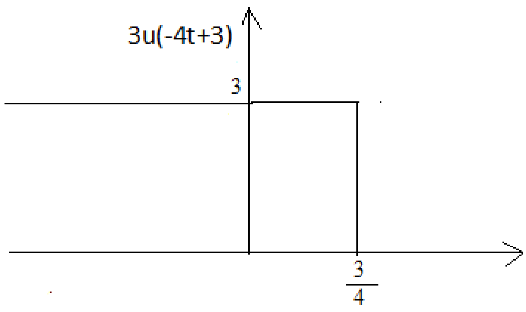


8. **Draw u(2t+1) using scaling and shifting operations.**



(always do scaling first, then shifting. In this case since it's a unit step function, there is no effect on scaling.) ie u(2(t + $\frac{1}{2}$))

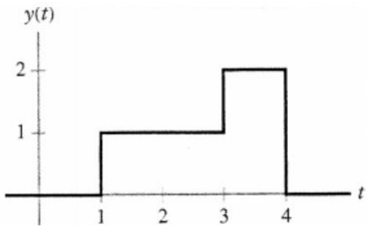
9. Draw 3u(-4t+3) using reversal, scaling and shifting operations.



(always do reversal first, then scaling, then shifting. ie 3u(-4(t - $\frac{3}{4}$))

UNSOLVED PROBLEMS

1. According to the product property of impulse response x(t) δ(t-to) = x(t0) δ(t-to), then what is the value of
   a) cost δ(t- $\pi/4$)
   b) (t^2 + 1) . δ(t-2)
   c) t . δ(t)
2. Prove the sampling property $\int_{t_1}^{t_2} \mathrm{x(t)}$. δ (t-t0)dt = x(t0) ; t1 ≤ t0 ≤ t2
3. According to the sampling property of impulse response
   $\int_{t_1}^{t_2} \mathrm{x(t)}$. δ (t-to)dt = x(t0) ; t1 ≤ to ≤ t2, then what is the value of
   a) $\int_0^{\infty} t^2 + 1.$ δ (t+1)dt
   b) $\int_0^{\infty} t^2 + 1.$ δ (t)dt
   c) $\int_0^{2\pi}$ sint. δ (π/2-t)dt
   d) $\int_2^4 e^{-(2t-1)}$. δ (3t-9)dt
4. Find the convolution of x(t) = e^{-2t} u(t) and Y(t)= u(t-3)
5. For the given signal, sketch



   a. y(t) – y(t-1)
   b. y(-2t)
   c. y(t-1)u(1-t)
   d. y(2t) + y(-3t)
   e. y(3t-1)
6. Determine if the following signals are power signals or energy signals
   a. x(t)=sin(2t) u(t)
   b. x(t)=tu(t)
   c. x(t)=5e^{-3t} u(t)
7. Sketch the even and odd part of the signal

8. Determine which of the following signals are bounded, and specify a smallest bound.
   a. x(t)=e^{3t} u(t)
   b. x(t)=4$e^{-6|t|}$

9. Determine if the following signals are periodic, and if so compute the fundamental period.
   a. $x[n] = e^{j\frac{20\pi}{3}n}$
   b. $x[n] = 1 + e^{j\frac{4\pi}{5}n}$

Revision 2021

Semester 5

SIGNALS & SYSTEMS

MODULE 2 NOTES

OUTCOMES & CONTENTS

Module Outcomes	Description	Duration (Hours)	Cognitive Level
CO2	**Classify and compare continuous time and discrete time systems**		
M2.01	Show time domain representation of a system	3	Understanding
M2.02	Compare continuous time and discrete time systems	3	Understanding
M2.03	Interpret impulse response of a continuous time and discrete time system.	3	Understanding
M2.04	Identify various properties of systems	3	Applying

Contents:

Representation of systems: Differential equation representation, Difference equation representation

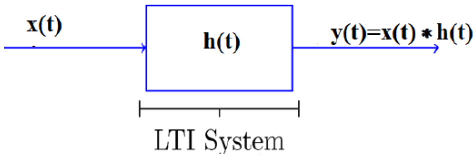
Continuous time and discrete time systems – Impulse response, examples

Properties of systems – linearity, time invariant system, invertible, casual and non-casual, stable and unstable.

CONTINUOUS TIME AND DISCRETE TIME SYSTEMS

CONTINUOUS TIME SYSTEM

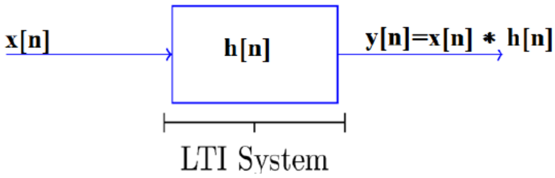
A continuous time system is a system in which continuous-time input signals are applied and result in continuous time output signals. Such a system can be shown as



Here x(t) is continuous-time time input and y(t) is continuous-time output. Examples of continuous-time system are electric circuits composed of resistors, capacitors and inductors that are driven by continuous-time sources. They are described by differential equations.

DISCRETE TIME SYSTEMS

A discrete-time system is a system that transform discrete-time input signals into discrete-time output signals



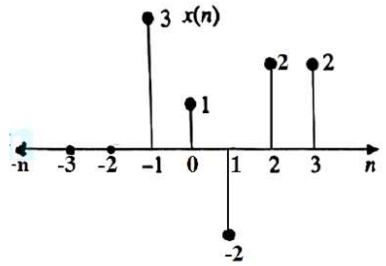
Here x[n] is the discrete-time input and y[n] is the discrete-time output. An example of discrete-time system is a simple model for the balance in a bank account from month-to-month. Discrete-time systems are described by difference equations.

COMPARISON OF CONTINUOUS AND DISCRETE TIME SYSTEMS

Continuous-time systems	Discrete-time systems
The input and output signal can be defined at any time instance and they can take all values in the continuous interval (a, b) where a can be -infinity and b can be +infinity	The input and output signal can be defined only at certain specific values of the time. These time instances need not be equidistant, but in practice, they are usually taken at equally spaced intervals.
They are described by differential equations	They are described by difference equations
The impulse response is denoted as h(t)	The impulse response is denoted as h[n]
Ex.: audio, video amplifiers, power supplies etc.	Ex.: microprocessors, semiconductor memories, shift registers etc.

REPRESENTATION OF DISCRETE TIME SIGNALS IN TERMS OF IMPULSES

- Consider the discrete time signal given by $x[n] = \left\{3, \underset{\uparrow}{1}, -2,2,2\right\}$
- This signal can be drawn as follows:



- The signal can also be represented in terms of impulses:

$x\,[n]$= 3. δ$[n + 1]$ + 1. δ $[n]$ – 2.δ $[n – 1]$ + 2. δ $[n – 2]$ + 2. δ $[n – 3]$

= $x[-1]$ δ$[n+1]$ + $x[0]$ δ $[n]$ + $x[1]$δ $[n – 1]$ + $x[2]$δ $[n – 2]$ + $x[3]$δ $[n – 3]$

Therefore, in general

$x\,[n]$= …+ $x[-1]$ δ$[n+1]$ + $x[0]$ δ $[n]$ + $x[1]$δ $[n – 1]$ + $x[2]$δ $[n – 2]$ + $x[3]$δ $[n – 3]$+…

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

where x [k] can be taken as weights.

Discrete Time Impulse Response and Convolution Sum

- The output of system is given by $y[n] = T\{x[n]\}$
- In an LTI discrete time system**,**

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\,T\,\{\delta[n-k]\}$$

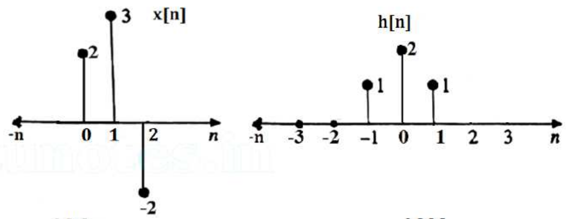
$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = x[n] * h[n]$$

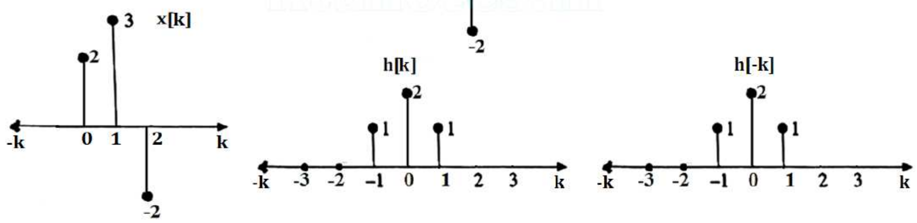
- This is called the convolution sum and h[n] is the impulse response of the system.
- h[n-k] is the response of the system to time shifted impulse.
- Hence if we know the impulse response of the LTI system, then we can find the response of the system to any other input.

Example:

What is the output of an LTI system with impulse response $h[n] = \{1, \underset{\uparrow}{2}, 1\}$ to the input x[n]={2,3,-2}? Use graphical method.

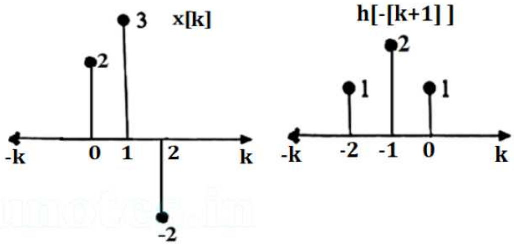
$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



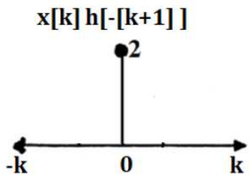


y[n] will range from n= -1 to n=3 (sum of lower limits of x[n] & h[n] to sum of upper limits of x[n] & h[n])

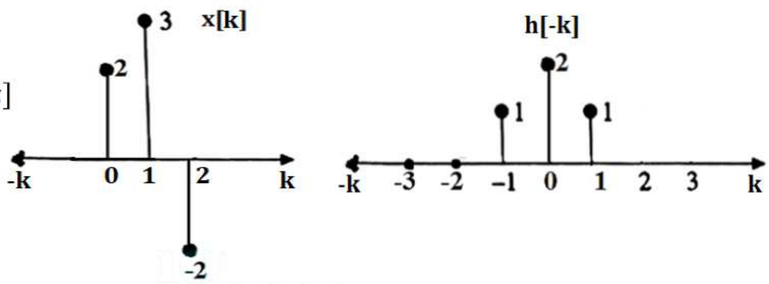
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

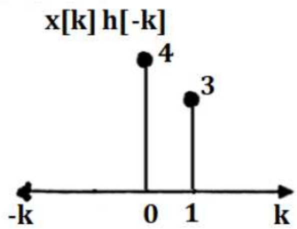


$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]\, h[-1-k]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[-[k+1]]$$
$$= 2$$

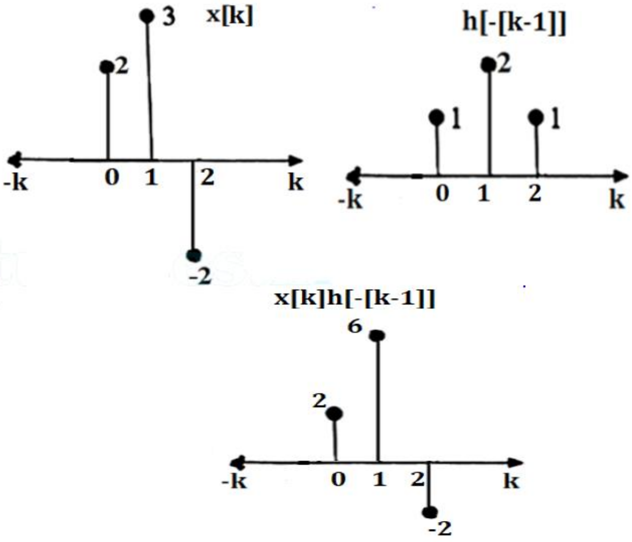


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$
$$= 7$$

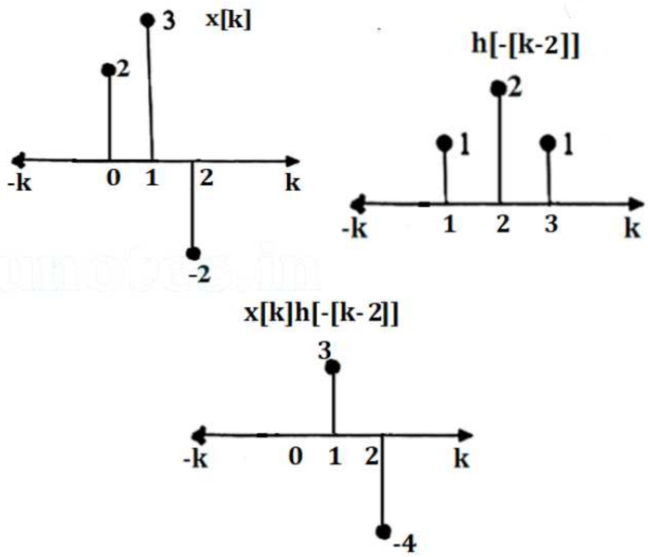




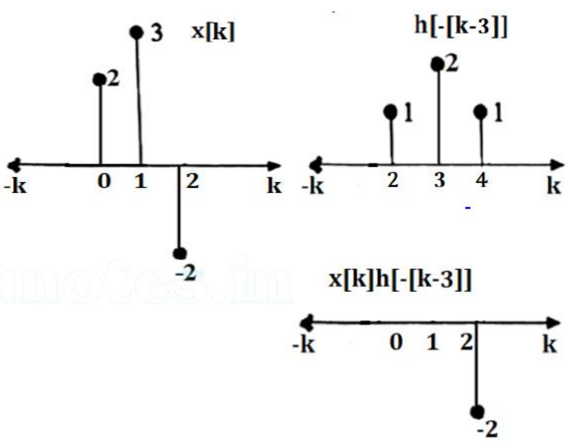
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[-[k-1]]$$
$$= 6$$



$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[-[k-2]]$$
$$= -1$$

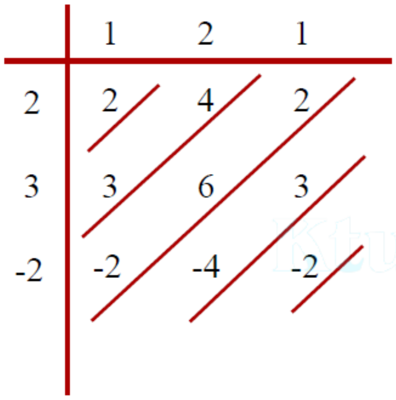


$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\, h[3-k]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[-[k-3]]$$
$$= -2$$



$$y[n] = \{2, \underset{\uparrow}{7}, 6, -1, -2\}$$

Number of samples in y[n] or length of linear convolution=L(length of x[n]) +M(length of h[n])-1 = 3+3-1=5



y[n]= {2,7,6,-1,-2}

REPRESENTATION OF CONTINUOUS TIME SIGNALS IN TERMS OF IMPULSES

- As discussed in the case of discrete time LTI systems, a continuous time signal can also be expressed in terms of impulses.
- In general

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Continuous Time Impulse Response and Convolution Sum

- The output of system is given by $y(t) = T\{x(t)\}$
- In an LTI continuous time system,

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\}$$

$$y(t) = \int_{-\infty}^{\infty} T\{\, x(\tau)\delta(t-\tau)\, d\tau\} = \int_{-\infty}^{\infty} x(\tau)\, T\{\delta(t-\tau)\}\, d\tau$$

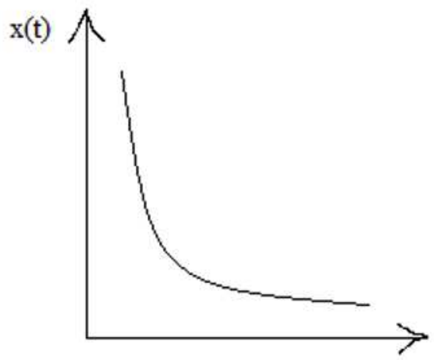
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

- This is called the convolution integral and h(t) is the impulse response of the system.
- Hence if we know the impulse response of the LTI system, then we can find the response of the system to any other input.

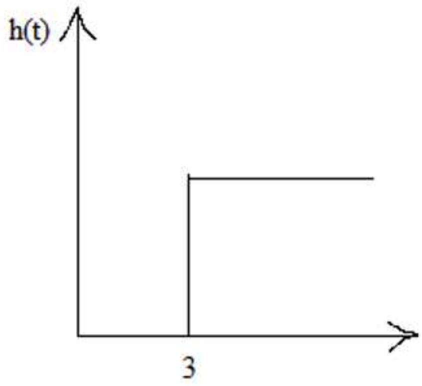
Example:

Find the output of a system whose impulse response is given by h(t)= u(t-3) for an input signal x(t) = e^{-2t} u(t).

x(t) = e^{-2t}**. u(t)**



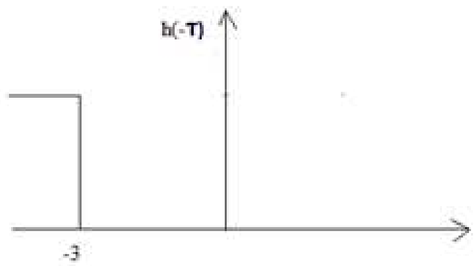
h(t)= u(t-3)

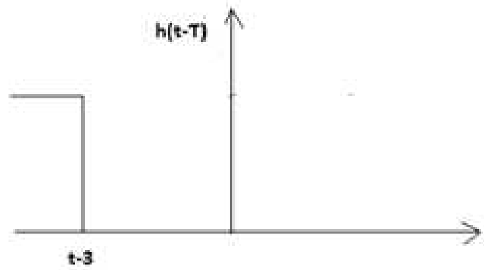


Change of axis

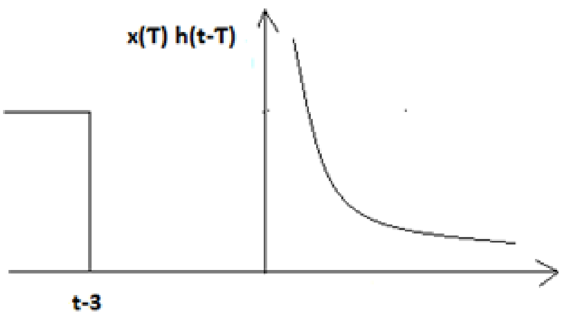
$x(t) = e^{-2t}u(t)$ and $h(t) = u(t-3)$ becomes

$x(\tau) = e^{-2\tau}u(\tau)$ and $h(\tau) = u(\tau - 3)$



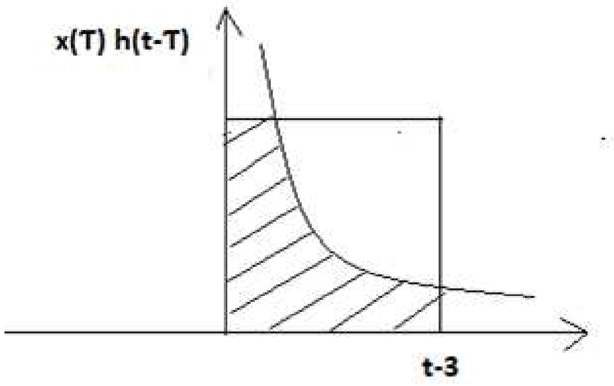


On sliding the input along the impulse response from -infinity to +infinity



Case 1 : t-3 < 0 (is not possible since 3<t<∞)

Case 2 : t-3 > 0



So on integration

$$y(t) = \int_0^{t-3} x(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_0^{t-3} e^{-2\tau}d\tau = \left.\frac{e^{-2\tau}}{-2}\right|_0^{t-3} = \frac{1-e^{-2(t-3)}}{2}$$

So the output of the system is

$$y(t) = \begin{cases} 0 & ; \quad t < 3 \\ \dfrac{1-e^{-2(t-3)}}{2} & ; \quad t \geq 3 \end{cases}$$

REPRESENTATION OF SYSTEMS

DIFFERENTIAL EQUATION REPRESENTATION

The general representation of a continuous time LTI system is given by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \qquad a_k \ \& \ b_k \ \ are\ constants$$

Differential equation provides an implicit specification of the system. The implicit expression describes a relationship between the input and the output rather than an explicit expression for the system output as a function of the input.

DIFFERENCE EQUATION REPRESENTATION

The general representation of a discrete time LTI system is given by the difference Equation

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]\ , \qquad a_k \ \& \ b_k \ \ are\ constants$$

Example:

Obtain the output of an LTI causal discrete- time system described by difference equation

$y[n] - \frac{1}{5}y[n-1] = x[n]$ to the input $x[n] = k\delta[n]$

Also calculate the impulse response.

Rewrite the difference equation in the form

$$y[n] = x[n]+\frac{1}{5}\,y[n-1]$$

To calculate the present output y[n], we need the past value of the output y[n-1]. But the system is causal that means there is no output before applying the input. Since

$$x[n] = K\ \delta[n]$$

i.e., x[n]=0 for n ≤ -1 implies that y[n] = 0 for n ≤ -1, so that we have as an initial condition y[- 1] = 0. Thus, to begin the recursion, with this initial condition, we can solve for successive values of y[n] for n ≥ 0 as follows:

$y[0] = x[0] + \frac{1}{5}\ y(-1) = K + 0 = K$ (since δ [n] = 1; if n = 0)

$y[1] = x[1] + \frac{1}{5}\ y(0) = 0 + \frac{1}{5}\ K = \frac{1}{5}\ K$

$y[2] = x[2] + \frac{1}{5}\ y(1) = 0 + \frac{1}{5}.\frac{1}{5}\ K = (\frac{1}{5})^2\ K$

$y[n] = x[n] + \mathbf{\frac{1}{5}}\ y[n-1] = = (\frac{1}{5})^2\ K$

Since this system is an LTI, its input-output behaviour is completely characterized by its impulse response.

If K = 1 or x[n] = δ [n], the output y[n] becomes the impulse response, i.e.,

$$h[n]= (\frac{1}{5})^2\ u[n]$$

PROPERTIES OF SYSTEMS

Systems are classified into the following categories:

1. Linear and Non-linear Systems
2. Time Variant and Time Invariant Systems
3. Linear Time variant and Linear Time invariant Systems
4. Static and Dynamic Systems
5. Causal and Non-causal Systems
6. Invertible and Non-Invertible Systems
7. Stable and Unstable Systems

1. Linear and Non-linear Systems

A linear system possesses the important property of super position: if an input consists of weighted sum of several signals, the output is also weighted sum of the responses of the system to each of those input signals. Mathematically let {y1[n]} be the response of the system to the input {x1[n]} and let {y2[n]} be the response of the system to the input {x2[n]}. Then the system is linear if:

1. Additivity: The response to $\{x_1[n]\} + \{x_2[n]\}$ is $\{y_1[n]\} + \{y_2[n]\}$

2. Homogeneity: The response to $a\{x_1[n]\}$ is $a\{y_1[n]\}$, where a is any real number if we are considering only real signals and a is any complex number if we are considering complex valued signals. A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T[a_1x_1(t) + a_2x_2(t)] = a_1T[x_1(t)] + a_2T[x_2(t)]$$

Hence,

$$T[a_1x_1(t) + a_2x_2(t)] = a_1y_1(t) + a_2y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example:

$y(t) = x_2(t)$

Solution:

$y_1(t) = T[x_1(t)] = x_1^2(t)$

$y_2(t) = T[x_2(t)] = x_2^2(t)$

$T [a_1x_1(t) + a_2x_2(t)] = [a_1x_1(t) + a_2x_2(t)]2$

Which is not equal to $a_1y_1(t) + a_2y_2(t)$. Hence the system is said to be non-linear.

2. Time Variant and Time Invariant Systems

A system is said to be time invariant if the behaviour and characteristics of the system do not change with time. Thus, a system is said to be time invariant if a time delay or time advance in the input signal leads to identical delay or advance in the output signal. Mathematically if

$$\{y[n]\} = T\{x[n]\}$$

Then

$$\{y[n-n_0]\} = T(\{x[n-n_0]\})$$

The condition for time invariant system is:

y(n, t) = y(n-t)

The condition for time variant system is:

y (n, t) ≠ y(n-t)

where y (n, t) = T[x(n-t)] = input change

y(n-t) = output change

Example 1:

y(n) = x(-n)

y(n, t) = T[x(n-t)] = x(-n-t)

y(n-t) = x(-(n-t)) = x(-n + t)

∴ y(n, t) ≠ y(n-t). Hence, the system is time variant.

Example 2:

y(n)=nx(n)

y(n, t)= T[x(n-t)]=nx(n-t)

y(n-t)=(n-t)x(n-t)

∴ y(n, t) ≠ y(n-t). Hence, the system is time variant.

3. Linear Time variant (LTV) and Linear Time Invariant (LTI) Systems

If a system is both linear and time variant, then it is called linear time variant (LTV) system. If a system is both linear and time Invariant then that system is called linear time invariant (LTI) system.

4. Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system. A system is said to be memory less if the output for each value of the independent variable at a given time n depends only on the input value at time n. For example, system specified by the relationship y[n] = cos(x[n]) + z is memory less. A particularly simple memory less system is the identity system defined by y[n] = x[n] In general we can write input-output relationship for memory less system as y[n] = g(x[n]). Not all systems are memory less. A simple example of system with memory is a delay defined by y[n] = x[n − 1]. A system with memory retains or stores information about input values at times other than the current input value.

Example 1:

y(t) = 2x(t)

For present value t=0, the system output is y(0) = 2x(0). Here, the output is only dependent upon present input. Hence the system is memory less or static.

Example 2:

y(t) = 2 x(t) + 3 x(t-3)

For present value t=0, the system output is y(0) = 2x(0) + 3x(-3).

Here x(-3) is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

5. Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input. $y[n] = f(x[n], x[n-1], \dots)$. All memory less systems are causal.

For non-causal system, the output depends upon future inputs also.

Example 1:

y(n) = 2 x(t) + 3 x(t-3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2).

Here, the system output only depends upon present and past inputs. Hence, the system is causal.

Example 2:

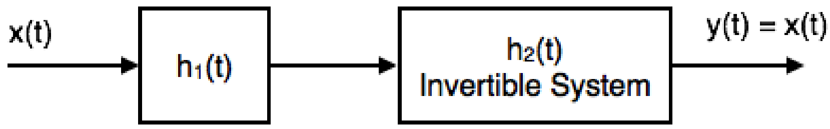
y(n) = 2 x(t) + 3 x(t-3) + 6x(t + 3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2) + 6x(4) Here, the system output depends upon future input. Hence the system is non-causal system.

For real time system where n actually denoted time causalities is important. Causality is not an essential constraint in applications where n is not time, for example, image processing. If we case doing processing on recorded data, then also causality may not be required.

6. Invertible and Non-Invertible systems

A system is said to be invertible if the input signal {x[n]} can be recovered from the output signal {y[n]}. For this to be true two different input signals should produce two different outputs. If some different input signal produce same output signal then by processing output we cannot say which input produced the output.



If y(t) = x(t), the system is invertible and if y(t) ≠ x(t), then the system is said to be non-invertible.

Example 1

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Then $x[n] = y[n] - y[n-1]$. Hence it is an invertible system.

Example if a non-invertible system is $y[n] = 0$. That is the system produces an all zero sequence for any input sequence. Since every input sequence gives all zero sequence, we cannot find out which input produced the output. The system which produces the sequence {x[n]} from sequence {y[n]} is called the inverse system. In communication system, decoder is an inverse of the encoder.

7. Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system, then it is said to be unstable.

A system is said to be BIBO stable if every bounded input produces a bounded output. We say that a signal {x[n]} is bounded if

$$|x[n]| < M < \infty$$

Note: For a bounded signal, amplitude is finite.

Example 1:

$y(t) = x^2(t)$

Let the input is u(t) (unit step bounded input) then the output $y(t) = u^2(t) = u(t)$ = bounded output. Hence, the system is stable.

Example 2: y (t) = ∫x(t)dt

the input is u (t) (unit step bounded input) then the output y(t) = ∫u(t)dt= ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when t → infinite). Hence, the system is unstable.

Example 3:

Consider a moving average system. $y[n] = \frac{1}{2N}\sum_{n=-N}^{N} x[n]$.

This is stable as y[n] is sum of finite numbers and so it is bounded.

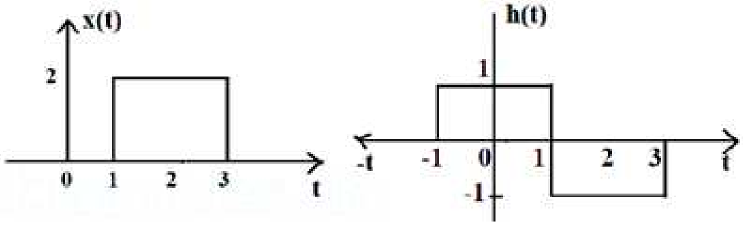
Example 4:

Consider $y[n] = \frac{1}{2N}\sum_{n=-\infty}^{\infty} x[n]$

This system is unstable since if we take $\{x[n]\} = \{u[n]\}$, the unit step then $y[0] = 1, y[1] = 2, y[2] = 3$, are $y[n] = n + 1, n \geq 0$, so y[n] grows without bound.

UNSOLVED PROBLEMS

1. Determine if each of the following systems is causal, memoryless, time invariant, linear, or stable. Justify your answer.
   a. $y(t) = 2x(t) + 3x^2(t-1)$
   b. $y(t) = cos^2(t)x(t)$
   c. $y(t) = x(-t)$
   d. $y(t) = x(3t)$
2. Determine if each of the following systems is causal, memoryless, time invariant, linear, or stable. Justify your answer.
   a. $y[n] = 3x[n]x[n-1]$
   b. $y[n] = 4x[3n-2]$
   c. $y[n] = \sum_{k=n-2}^{n+2} x[k]$
3. Using the graphical method, compute and sketch $y[n] = x[n] * h[n]$ for
   a. $x[n] = u[n]$ and $h[n] = (1/2)^n u[n-1]$
   b. $x[n] = 1$ and $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$
   c. $x[n] = u[n-1] - u[n-3]$ and $h[n] = -u[n] + u[n-3]$
4. Using the graphical method, compute and sketch $y[n] = x[n] * h[n]$ for
   a. $h(t) = e^{-t}u(t)$ and $x(t) = 2u(t) - 2u(t-1)$
   b. $h(t) = e^{-|t|}$ and $x(t) = u(t)$
   c. $h(t) = e^{t}u(-t)$ and $x(t) = u(t-2)$
5. Suppose the input to LTI system is x(t) and impulse response is h(t), find the output of the system



Revision 2021

Semester 5

SIGNALS & SYSTEMS

MODULE 3 NOTES

OUTCOMES & CONTENTS

Module Outcomes	Description	Duration (Hours)	Cognitive Level
CO3	**Explain Fourier representation of signals**		
M3.01	Apply Fourier series and discrete time Fourierseries	5	Applying
M3.02	Summarize the properties of Fourier series	3	Understanding
M3.03	Explain sampling theorem, aliasing,reconstruction	4	Understanding
M3.04	Apply Fourier transform, discrete time Fourier transform	4	Applying

Contents:

Fourier representation of four class of signals

- Continuous time periodic signal: Fourier series (FS)
- Discrete time periodic signal: Discrete time Fourier series (DTFS)
- Continuous time non-periodic signal: Fourier transform (FT)
- Discrete time non-periodic signal: Discrete time Fourier transform (DTFT)

Properties of Fourier representation – linearity, symmetry, time shift, frequency shift, scaling, differentiation and integration, convolution and modulation

Sampling theorem, aliasing, reconstruction

FOURIER REPRESENTATION OF SIGNALS

INTRODUCTION

The Fourier representation of a signal refers to expressing the signal in terms of its frequency components using Fourier analysis. It allows us to decompose a signal into a sum of sinusoidal functions of different frequencies, amplitudes, and phases. The basic idea behind the Fourier representation is that any periodic or non-periodic signal can be represented as a combination of sine and cosine functions with different frequencies. These sinusoidal functions are referred to as **Fourier components or Fourier harmonics.**

There are four distinct Fourier representations, each applicable to a different class of signals. These four classes are defined by the periodicity properties of a signal and whether it is continuous or discrete time:

- ❖ Periodic signals have Fourier series representations. The Fourier series (FS) applies to continuous-time periodic signals and the discrete time Fourier series (DTFS) applies to discrete time periodic signals.
- ❖ Non-periodic signals have Fourier transform representations. If the signal is continuous time and non-periodic, the representation is termed the Fourier transform (FT). If the signal is discrete time and non-periodic, then the representation is termed the discrete time Fourier transform (DTFT).

FOUR DISTINCT FOURIER REPRESENTATIONS

Time Property	**Periodic**	**Non-periodic**
Continuous (t)	**Fourier Series (FS)**	**Fourier Transform (FT)**
Discrete [n]	**Discrete-Time Fourier Series (DTFS)**	**Discrete-Time Fourier Transform (DTFT)**

The Fourier representation provides valuable information about the frequency content of a signal. By decomposing the signal into its frequency components, we can analyse its spectral content, identify dominant frequencies, filter out unwanted components, and manipulate signals in the frequency domain. It has widespread applications in fields such as signal processing, communications, audio and image processing, and many others.

FOURIER SERIES REPRESENTATION OF CONTINUOUS TIME PERIODIC SIGNALS (CTFS)

Any signal can be represented as a linear combination of some basis signals. As already discussed, continuous time signals can be represented as a linear combination of shifted and scaled impulses. Here the basis signal is an impulse. In a similar way, sinusoids or complex exponentials are commonly used as basic functions for representing signals in the frequency domain.

In the case of continuous-time signals, a signal can be decomposed into a sum of sinusoidal or complex exponential functions with different frequencies, amplitudes, and phases. This representation is known as the **Continuous Time Fourier series representation**. The Fourier series allows us to express a periodic signal as a linear combination of harmonic components.

A periodic signal is one which repeat itself periodically over $-\infty < t < \infty$.

i.e., A signal is periodic if, $x(t) = x(t+T)$ for all t

where, *T* is the fundamental period and $\omega_0 = 2\pi / T$ is referred to as the fundamental frequency.

The two basic periodic signals are the sinusoidal signal and the complex exponential signal.

$$x(t) = \cos(\omega_0 t)$$
$$x(t) = e^{jk\omega_0 t}$$

Both of these signals are periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi / T$. The Fourier series is a decomposition of such periodic signals into the sum of a (possibly infinite) number of complex exponentials whose frequencies are *harmonically* related.

The set of harmonically related complex exponentials is given by

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm1, +2, \ldots$$

Thus, a periodic signal $x(t)$ can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = 2\pi / T \text{ = Fundamental Frequency}$$

$c_n =$ fourier coefficients and $\pm n\omega_o =$ harmonic frequencies

The term for *n=0* is a constant
The terms for *n=±1* are the fundamental/first harmonic components
The terms for *n=±2* are the second harmonic components

The terms for $n=\pm N$ are the N^{th} harmonic components

COMPUTING FOURIER COEFFICIENTS

Multiply above equation by $e^{-jk\omega_0 t}$ and integrate over one period.

Then,

$$\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt = \int_{t_0}^{t_0+T}\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right]e^{-jk\omega_0 t}dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t}e^{-jk\omega_0 t}dt$$

Substituting the relation $\int_{t_0}^{t_0+T} e^{jn\omega_0 t}e^{-jk\omega_0 t}dt = \begin{cases}0, & k \neq n \\ T, & k = n\end{cases}$ in the above equation we get,

$$\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt = Tc_k$$

Hence,

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt$$

Or

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$$

Where c_n are the Fourier series coefficients or spectral coefficients of $x(t)$

Thus, the Fourier series of a continuous time periodic signal is given by the following equations:

$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jk\omega_0 t} = \sum_{n=-\infty}^{+\infty} c_n e^{jk(2\pi/T)t}$	**Synthesis Equation**
$c_n = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jk(2\pi/T)t}dt$	**Analysis Equation**

FREQUENCY SPECTRUM OF CONTINUOUS TIME PERIODIC SIGNALS

The exponential representation of a periodic signal $x(t)$ contains amplitude coefficients c_n which are complex. i.e., The Fourier coefficient, is a complex quantity and so it can be expressed as:

$$c_n = |c_n| \angle c_n$$

Where, $|c_n|$ = magnitude of c_n and $\angle c_n$ = phase of c_n

The term, $|c_n|$ represents the magnitude of nth harmonic component and term $\angle c_n$ denotes the phase of n^{th} harmonic component. Therefore, we can plot two spectra, the magnitude spectrum ($|c_n|$ *versus* n) and phase spectrum (*angle* c_n *versus* n). The plot of harmonic magnitude/phase of a signal versus 'n' is called frequency spectrum (or Line spectrum). The plot of harmonic magnitude versus 'n' is called magnitude spectrum and the plot of harmonic phase vs 'n' is called phase spectrum. The two plots together are known as Fourier frequency spectra of $x(t)$. It is also known as frequency domain representation. The Fourier spectrum exists only at discrete frequencies $n\omega_0$ where n = 0, 1, 2 ... hence it is also known as discrete spectrum.

The spectra can be plotted for both positive & negative frequencies. Hence it is called two-sided spectra. The magnitude spectrum is symmetrical about the vertical axis passing through the origin and phase spectrum is anti-symmetrical about the vertical axis passing through origin. So, magnitude spectrum exhibits even symmetry & phase spectrum exhibits odd symmetry.

POWER CONTENT OF A PERIODIC SIGNAL

The average power of a periodic signal $x(t)$ over any period is

$$P = \frac{1}{T}\int_T |x(t)|^2 \, dt$$

If $x(t)$ is represented by the complex exponential Fourier series, then we can compute the power of the signal from the Fourier coefficients:

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{n=-\infty}^{+\infty} |c_n|^2$$

This equation is called *Parseval's identity (or* ***Parseval's Theorem****)* for the Fourier series. The significance of this result is that the power of a signal is directly dependent on the magnitude square of its Fourier coefficients. Hence, the energy or power of a signal in the time domain is equal to the energy or power in the frequency domain.

CONDITIONS FOR EXISTENCE OF FOURIER SERIES

Every signal $x(t)$ of period *T* satisfying following conditions known as Dirichlet's conditions, can be expressed in the form of Fourier series

1. The signal should have only a finite number of maxima and minima over a given period.
2. The signal must possess only a finite number of discontinuities over a given period.
3. The signal must be absolutely integrable over a given period i.e.,

$$\int_0^T |x(t)|\,dt < \infty$$

It guarantees that each coefficient c_k will be finite, i.e.,

$$|c_k| < \infty$$

<u>Example 1</u>.

Consider a periodic signal $x(t)$, with fundamental frequency 2π that is expressed in the form of equation as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

Where,

$$a_0 = 1,$$
$$a_1 = a_{-1} = \frac{1}{4},$$
$$a_2 = a_{-2} = \frac{1}{2},$$
$$a_3 = a_{-3} = \frac{1}{3}.$$

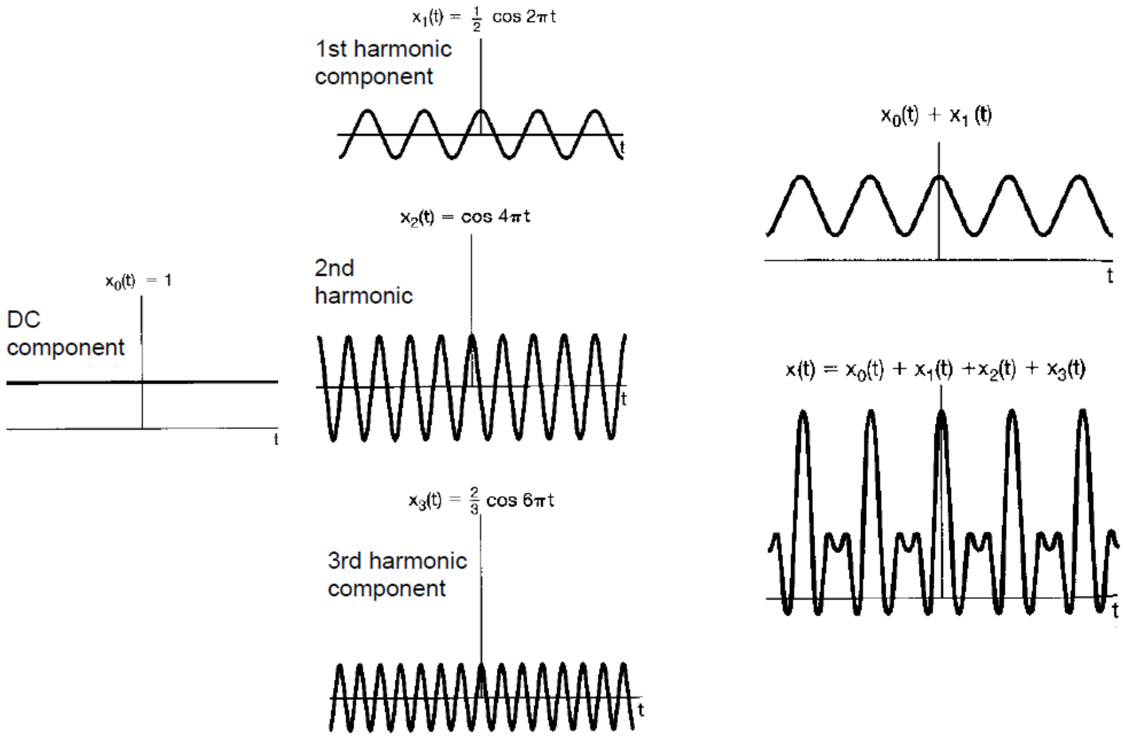
Rewriting the equation and collecting each of the harmonic components which have the same frequency, we obtain

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

Equivalently, using Euler's relation, we can write $x(t)$ in form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

Figure below gives the graphical illustration of how the signal $x(t)$ is built up from its harmonic components



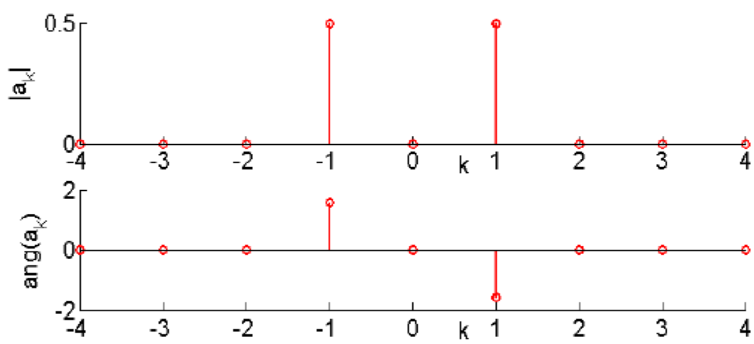
Example 2.

Find the Fourier Series of $\sin(\omega_0 t)$

The fundamental period of $\sin(\omega_0 t)$ is ω_0. By inspection we can write:

$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

So, $c_1 = \frac{1}{2j}$, $c_{-1} = -\frac{1}{2j}$ and $c_k = 0$ otherwise



PROPERTIES OF FOURIER SERIES

Consider a signal, $x(t)$ which is periodic with period, *T* and fundamental frequency, $\omega_0 = 2\pi / T$

Let the Fourier coefficients of $x(t)$ be denoted by C_n

Then,

$$x(t) \xleftrightarrow{FS} c_n$$

1. **Linearity**
   If $x(t) \xleftrightarrow{FS} c_n$ and $y(t) \xleftrightarrow{FS} d_n$
   Then, $ax(t) + by(t) \xleftrightarrow{FS} ac_n + bd_n$
   i.e., Fourier Series is a linear operation.

2. **Time Shifting**
   If $x(t) \xleftrightarrow{FS} c_n$
   Then according to time shifting property,
   $x(t - t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} c_n$
   i.e., Magnitude of Fourier Series coefficients remains unchanged when the signal is shifted in time.

3. **Frequency Shifting**
   If $x(t) \xleftrightarrow{FS} c_n$
   Then according to frequency shifting property,
   $e^{jm\omega_0 t_0} x(t) \xleftrightarrow{FS} C_{(n-m)}$

4. **Time Scaling**
   If $x(t)$ is periodic with period T, then $x(at)$ will be periodic with period T / a; a>0
   If $x(t) \xleftrightarrow{FS} c_n$
   Then $x(at) \xleftrightarrow{FS} c_n$
   Thus, after time scaling FS coefficients are the same. But, the spacing between the frequency components changes from ω_0 to $a\omega_0$ or from $1/T$ to a/T

5. **Time Inversion**
   Time inversion property states that
   If $x(n) \xleftrightarrow{FS} c_k$
   Then $x(-t) \xleftrightarrow{FS} c_n^*$

6. Differentiation in Time

According to this property, if $x(t)\xleftrightarrow{FS}c_n$

Then $\frac{d}{dt}x(t)\xleftrightarrow{FS}(jn\omega_0)c_n$

7. Integration

If $x(t)\xleftrightarrow{FS}c_n$

Then $\int x(t)dt\xleftrightarrow{FS}\frac{1}{jn\omega_0}c_n$

8. Convolution

If $x(t)\xleftrightarrow{FS}c_n$ and $y(t)\xleftrightarrow{FS}d_n$

Then $x(t)*y(t)\xleftrightarrow{FS}Tc_nd_n$

Hence, the convolution in time domain leads to multiplication of Fourier series coefficients in Fourier series domain.

9. Multiplication in Time Domain

If $x(t)\xleftrightarrow{FS}c_n$ and $y(t)\xleftrightarrow{FS}d_n$

We have $x(t)y(t)\xleftrightarrow{FS}\sum_{m=-\infty}^{\infty}c_md_{n-m}$

Multiplication in time domain leads to convolution in Fourier series domain.

10. Symmetry

Symmetry properties state that

If $x(t)$ is real, then $\rightarrow c_n = c_{-n}^*$

If $x(t)$ is imaginary, then, $\rightarrow c_n = -c_{-n}^*$

FOURIER SERIES REPRESENTATION OF DISCRETE TIME PERIODIC SIGNALS (DTFS)

As with continuous time signals, a periodic discrete time signal with fundamental period *N* can be expressed as a series combination of *N* harmonically related complex exponentials. The summation of the frequency components gives the Fourier series representation of periodic discrete time signal, where the discrete time signal is represented as a function of frequency ω. The Fourier series of discrete time signal is called **Discrete Time Fourier Series (DTFS).** The frequency components are also called frequency spectrum of the discrete time signal. Unlike FS representation of continuous time signals, DTFS is finite.

Let $x(n)$ be a periodic signal with a fundamental period *N*, i.e.,

$$x(n) = x(n+N)$$

An *N*-periodic discrete-time signal can be expanded as,

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$
$$= \sum_{k=0}^{N-1} c_k e^{jk\omega_o n}$$

Where,

c_k = Fourier Coefficients

$\omega_0 = 2\pi / N$ = Fundamental frequency of $x(n)$

$k\omega_0$ = k^{th} harmonic frequency of $x(n)$

$c_k e^{jk\omega_o n}$ = k^{th} harmonic component of $x(n)$

The Fourier coefficients, c_k can be evaluated using the equation

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$
$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-jk\omega_o n} \qquad \text{for } k = 0,1,2,...,N-1$$

The Fourier coefficient c_k represents the amplitude and phase associated with the *k^th^* frequency component. Hence, we can say that the Fourier coefficients provide the description of $x(n)$ in the frequency domain.

COMPUTING FOURIER COEFFICIENTS

Multiply expression for $x(n)$ with $e^{-j2\pi mn/N}$ and sum over time *n* over one period:

$$\frac{1}{N}\sum_{n=0}^{N-1}\left[e^{-j2\pi mn/N}\times\left(x(n)=\sum_{k=0}^{N-1}c_k e^{j2\pi kn/N}\right)\right]$$

Here the right-hand side becomes

$$\frac{1}{N}\sum_{k=0}^{N-1}c_k\sum_{n=0}^{N-1}e^{j2\pi(k-m)n/N}$$

which vanishes if $m \neq k$, because

$$\sum_{n=0}^{N-1}e^{j2\pi(k-m)n/N} = \begin{cases} N, \text{ if } k - m = 0, \pm N, \pm 2N, \ldots \\ 0, \text{ otherwise} \end{cases}$$

Hence,

$$\frac{1}{N}\sum_{k=0}^{N-1}c_k\sum_{n=0}^{N-1}e^{j2\pi(k-m)n/N} = c_m$$

And

$$\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi mn/N} = c_m$$

Or,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N}$$

Thus, the Fourier series (DTFS) of a discrete time periodic signal is given by the following equations:

$x(t) = \sum_{k=0}^{N-1}c_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1}c_k e^{j2\pi kn/N}$	**Synthesis Equation**
$c_k = \frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-jk\omega_o n}$	**Analysis Equation**

DIFFERENCE BETWEEN CONTINUOUS TIME AND DISCRETE TIME FOURIER SERIES

1. The frequency range of continuous time signal is from $-\infty$ to $+\infty$, and so it has infinite frequency spectrum.
2. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has infinite frequency spectrum. A discrete time signal with fundamental period N will have *N* frequency components whose frequencies are,
   $\omega_k = 2\pi k / N$ for $k = 0,1,2,....,N-1$

FREQUENCY SPECTRUM OF DISCRETE TIME PERIODIC SIGNALS

The DTFS representation of a periodic signal $x(n)$ contains Fourier coefficients, c_n which are complex and can be expressed as:

$$c_k = |c_k| \angle c_k \quad ; \quad k = 0,1,2,....,N-1$$

Where, $|c_k|$ is the magnitude of c_k and $\angle c_k$ is the phase of c_k

The term, $|c_k|$ represents the magnitude of k^{th} harmonic component and term $\angle c_k$ denotes the phase of k^{th} harmonic component. The plot of harmonic magnitude/phase of a discrete time signal versus 'k' is called frequency spectrum. The plot of harmonic magnitude versus 'k' is called magnitude spectrum and the plot of harmonic phase vs 'k' is called phase spectrum.

The Fourier coefficients are periodic with period *N*

$$c_{k+N} = c_k$$

Since Fourier coefficients are periodic, we say frequency spectrum of DT periodic signal is periodic, with period *N*

PROPERTIES OF DISCRETE TIME FOURIER TRANSFORM

Consider a signal $x(n)$ which is periodic with period *N* and fundamental frequency $\omega_0 = 2\pi / N$ Let the Fourier coefficients of $x(n)$ be denoted by c_k

Then,

$$x(n) \xleftrightarrow{DTFS} c_k$$

1. **Linearity**
   If $x(n) \xleftrightarrow{DTFS} c_k$ and $y(n) \xleftrightarrow{DTFS} d_k$
   Then, $ax(n) + by(n) \xleftrightarrow{DTFS} ac_k + bd_k$
   i.e., Fourier Series is a linear operation.

2. Time Shifting

If $x(n) \xleftrightarrow{DTFS} c_k$

Then according to time shifting property,

$x(n-m) \xleftrightarrow{DTFS} e^{-jk\omega_0 m} c_k$

i.e., Magnitude of Fourier Series coefficients remains unchanged when the signal is shifted in time.

3. Frequency Shifting

If $x(n) \xleftrightarrow{DTFS} c_k$

Then according to frequency shifting property,

$e^{jm\omega_0 n} x(t) \xleftrightarrow{DTFS} c_{k-m}$

4. Time Scaling

If $x(n)$ is periodic with period *N*, then $x(n/m)$ (where *N* multiple of *m*) will be periodic with period *mN*

If $x(n) \xleftrightarrow{DTFS} c_k$

Then $x\left(\frac{n}{m}\right) \xleftrightarrow{FS} \frac{1}{m} c_k$

5. Time Reversal

Time inversion property states that

If $x(n) \xleftrightarrow{DTFS} c_k$

Then, $x(-n) \xleftrightarrow{DTFS} c_{-k}$

6. Multiplication

If $x(n) \xleftrightarrow{DTFS} c_k$ and $y(n) \xleftrightarrow{DTFS} d_k$

We have $x(n)y(n) \xleftrightarrow{DTFS} \sum_{m=0}^{N-1} c_m d_{k-m}$

Multiplication in time domain leads to convolution in Fourier series domain.

7. Convolution

If $x(n) \xleftrightarrow{DTFS} c_k$ and $y(n) \xleftrightarrow{DTFS} d_k$

We have $\sum_{m=0}^{N-1} x(m)y((n-m))_N \xleftrightarrow{DTFS} Nc_k d_k$

8. Symmetry

Symmetry properties state that

If $x(n)$ is real, then $c_k = c_{-k}^*$

If $x(n)$ is imaginary, then $c_k = -c_{-k}^*$

SAMPLING OF CONTINUOUS TIME (ANALOG) SIGNALS

The sampling is the process of conversion of a continuous time signal into a discrete time signal.

We can obtain a discrete-time signal by sampling a continuous-time signal at equally spaced time instants,

$$t = nT_s$$

$$x(n) = x(nT_s); \qquad -\infty < n < \infty$$

The individual values $x(n)$ are called the samples of the continuous time signal, $x(t)$.

Usually, the time interval between successive samples will be the same and such type of sampling is called periodic or uniform sampling. The time interval T_s between successive samples is called sampling period. The inverse of the sampling period is called sampling frequency or sampling rate and is denoted by f_s

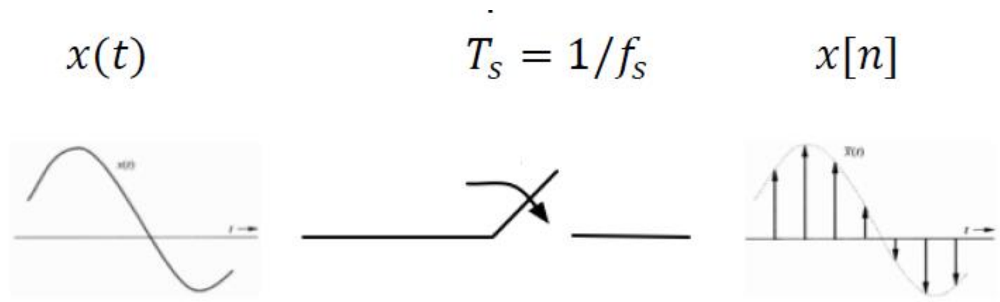
Mathematically, we can write it as,

$$f_s = 1/T_s$$

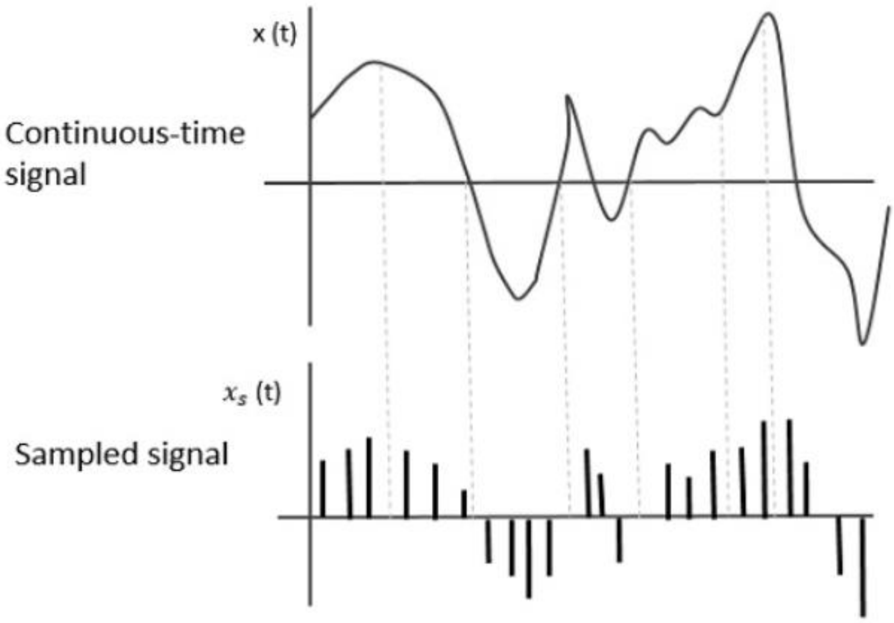
Where,

T_s = sampling period in seconds

f_s = sampling rate in Hertz



The following figure shows a continuous-time signal $x(t)$ and the corresponding sampled signal $x_s(t)$. When $x(t)$ is multiplied by a periodic impulse train, the sampled signal $x_s(t)$ is obtained.



SAMPLING THEOREM

A bandlimited continuous time signal with maximum frequency f_m can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component f_m of message signal. i. e.,

$$f_s \geq 2f_m$$

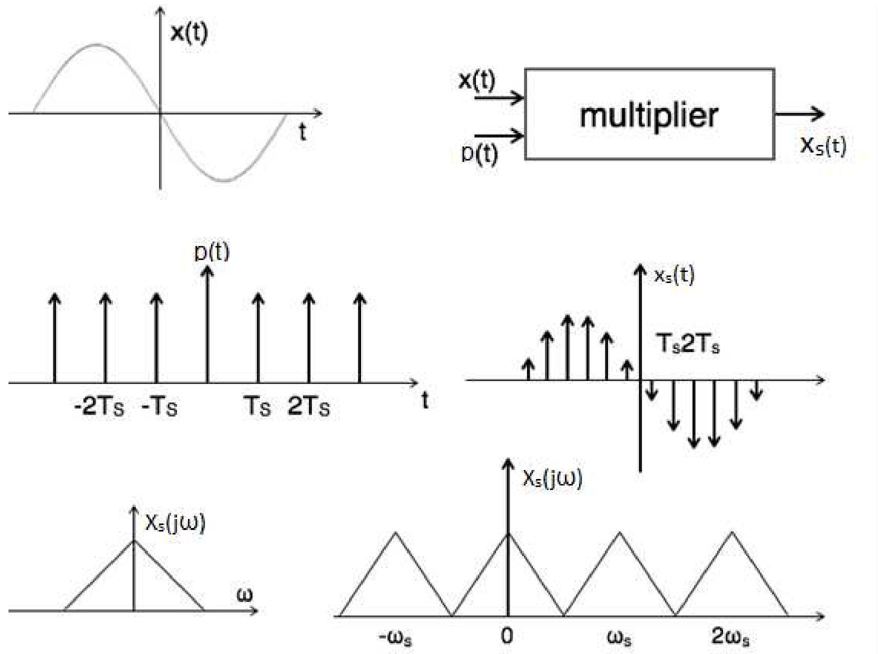
In other words, Bandlimited continuous time signals when sampled properly, can be represented as discrete-time signals with no loss of information. This remarkable result is known as the **Sampling Theorem**.

Sampling theorem is the bridge between continuous-time and discrete-time signals. It states how often we must sample in order not to lose any information.

NYQUIST RATE

When the sampling frequency f_s is equal to twice the maximum frequency of the given signal, the sampling rate is called **Nyquist rate**. It is the minimum sampling frequency needed to reconstruct the analog signal from sampled waveform. The corresponding sampling interval $T_s = \frac{1}{2f_m}$ is called the Nyquist interval

Proof: Consider a continuous time signal $x(t)$. The spectrum of $x(t)$ is a band limited to f_m Hz i.e. the spectrum of $x(t)$ is zero for $|\omega| > \omega_m$.



Sampling of input signal $x(t)$ can be obtained by multiplying $x(t)$ with an impulse train $p(t)$ of period T_s. The output of multiplier is a discrete signal called sampled signal which is represented with $x_s(t)$ in the following diagrams:

Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

The expression for a pulse train is given by,

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampled version of the continuous time signal, $x(t)$ is

$$x_s(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

We can derive the complex Fourier series of a pulse train:

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} \text{ where } \omega_s = 2\pi / T_s$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}dt = \frac{1}{T}\left[e^{-jk\omega_s t}\right]_{t=0} = \frac{1}{T}$$

$$p(t)=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_s t}$$

The Fourier series of the sampled signal, $x_s(t)$ is:

$$x_s(t)=p(t)x(t)=\sum_{k=-\infty}^{\infty}\frac{1}{T}x(t)e^{jk\omega_s t}$$

Recalling the Fourier transform properties of linearity (the transform of a sum is the sum of the transforms) and modulation (multiplication by a complex exponential produces a shift in the frequency domain), we can write an expression for the Fourier transform of our sampled signal:

$$X_s(j\omega)=FT\left\{x_s(t)\right\}$$

$$=FT\left\{p(t)x(t)\right\}=FT\left\{\sum_{k=-\infty}^{\infty}\frac{1}{T}x(t)e^{jk\omega_s t}\right\}$$

$$=\frac{1}{T}\sum_{k=-\infty}^{+\infty}FT\left\{x(t)e^{jk\omega_s t}\right\}$$

$$=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X(j(\omega-k\omega_s))$$

Thus, the Fourier transform of the sampled signal is given by an infinite sum of frequency shifted and amplitude scaled replicas of the spectrum of original continuous time signal.

Since the original signal, $x(t)$ is bandlimited, the highest frequency component present in it is f_m.

i.e.,

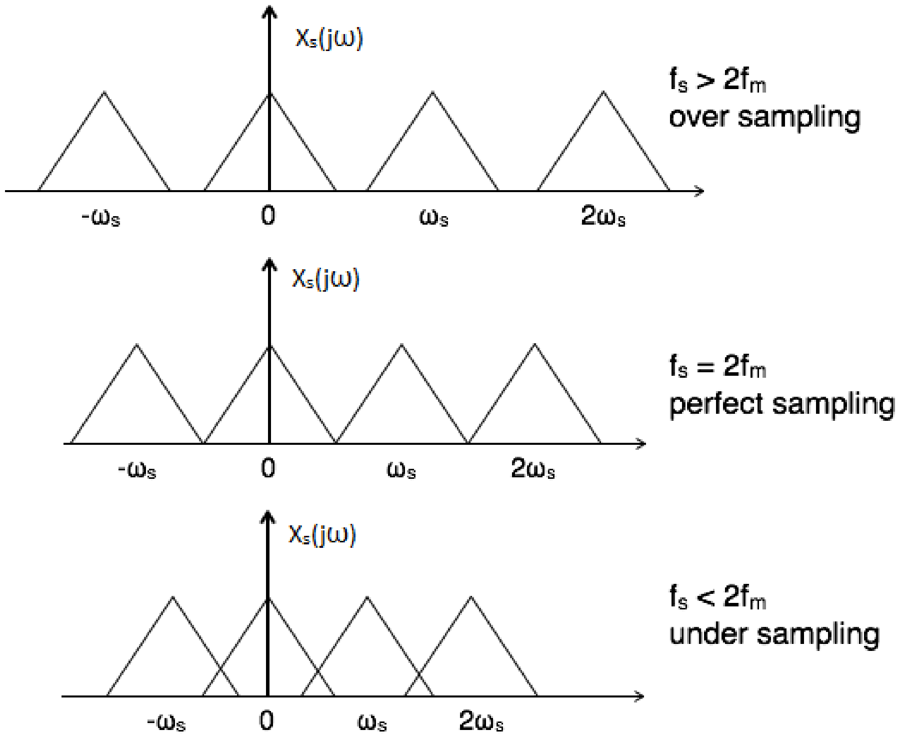
$$X(j\omega)=0;\ \text{for } |\omega|>\omega_m$$

To reconstruct $x(t)$, we must recover input signal spectrum $X(\omega)$ from sampled signal spectrum $X_s(j\omega)$, which is possible when there is no overlapping between the cycles of $X_s(j\omega)$.

Depending on the relation between maximum frequency content, f_m of the original signal and the sampling frequency f_s, we have two cases:

1. $f_s \geq 2f_m$: The replicas of original spectrum do not overlap and the signal can be reconstructed from the sample spectrum. Here the signal is perfectly sampled without any information loss.

2. $f_s < 2f_m$: The replicas of original spectrum overlap leading to mixing up and loss of information. This unwanted phenomenon of overlapping is called as Aliasing



ALIASING

Aliasing can be referred to as "the phenomenon of a high-frequency component in the spectrum of a signal, taking on the identity of a low-frequency component in the spectrum of its sampled version."

The corrective measures taken to reduce the effect of Aliasing are:

- A **low pass anti-aliasing filter** is employed, before the sampler, to eliminate the high frequency components, which are unwanted.
- The signal which is sampled after filtering, is sampled at a rate slightly higher than the Nyquist rate. i.e., $f_s > 2f_m$

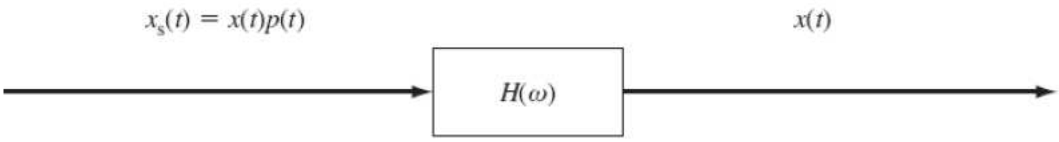
SIGNAL RECONSTRUCTION

When the spectrum of sampled signal has no aliasing then it is possible to recover the original signal from the sampled signal.

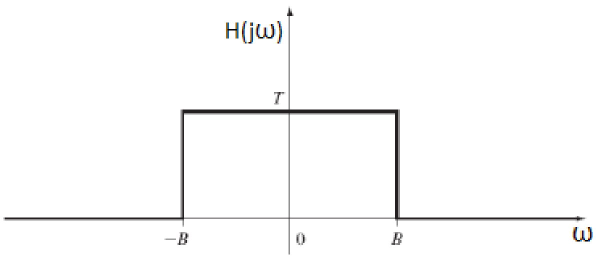
In order to reconstruct the original signal $x(t)$, we can use an ideal lowpass filter on the sampled spectrum which has a bandwidth B of any value between f_m and $(f_s - f_m)$. The filter will pass only the portion of sampled spectrum, $X_s(f)$, centred at $f = 0$ and will reject all its replicas at $f = nf_s$, for $n \neq 0$. This implies that the shape of the continuous time signal $x_s(t)$, will be retained at the output of the ideal filter.

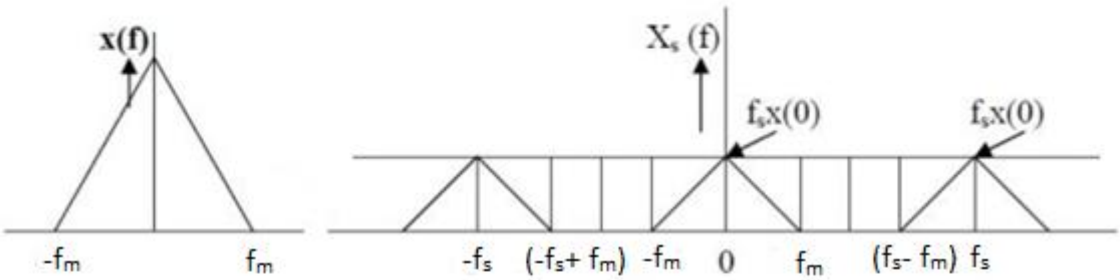
The frequency response of the lowpass filter is:

$$H(j\omega) = \begin{Bmatrix} T, & -B \leq \omega \leq B \\ 0, & \text{elsewhere} \end{Bmatrix}$$



Lowpass filter





Reconstruction process is possible only if the shaded parts do not overlap. This means that f_s must be greater that twice f_m.

Properties of Continuous Time Fourier Series		
Property	Periodic Signal	Fourier Transform
	$\begin{cases} x(t) \\ y(t) \end{cases}$ *Periodic with Period T and fundamental frequency* $\omega_0 = \frac{2\pi}{T}$	$\begin{cases} a_k \\ b_k \end{cases}$
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(\frac{2\pi}{T})t_0}$
Frequency Shifting	$e^{jM\omega_0 t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(at),\quad a > 0$ (*Periodic with period* $\frac{T}{a}$)	a_k
Periodic Convolution	$x(t) * y(t) = \int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_o a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$ (finite value and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_o}\right)a_k = \left(\frac{1}{jk\frac{2\pi}{T}}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ *is real*	$\begin{cases} a_k = a_{-k}^* \\ Re\{a_k\} = Re\{a_{-k}\} \\ Im\{a_k\} = -Im\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ *is real and even*	a_k *is real and even*
Symmetry for Real and Odd Signals	$x(t)$ *is real and odd*	a_k *is purely inmaginary and odd*
Even-Odd Decomposition for Real Signals	$x_r(t) = Ev\{x(t)\}\quad [x(t) real]$ $x_o(t) = Od\{x(t)\}\quad [x(t) real]$	$Re\{a_k\}$ $jIm\{a_k\}$
Parseval's Relation for Periodic Signals	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	

Properties of Discrete Time Fourier Series

Property	Periodic Signal	Fourier Transform
$\begin{cases} x[n] \\ y[n] \end{cases}$ *Periodic with Period N and fundamental frequency* $\omega_0 = \frac{2\pi}{N}$		$\begin{cases} a_k \\ b_k \end{cases}$ *Periodic with Period N*
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(\frac{2\pi}{N})n_0}$
Frequency Shifting	$e^{jM(\frac{2\pi}{N})n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x\left[\frac{n}{m}\right] & if\ n\ is\ a\ multiple\ of\ m \\ 0 & if\ n\ is\ not\ a\ multiple\ of\ m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic With period mN)
Periodic Convolution	$\sum_{r=<N>} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk2\pi/N})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ Finite valued and periodic only if $a_0 = 0$	$\left(\frac{1}{\left(1 - e^{-\frac{jk2\pi}{N}}\right)}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ *is real*	$\begin{cases} a_k = a_{-k}^* \\ Re\{a_k\} = Re\{a_{-k}\} \\ Im\{a_k\} = -Im\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Symmetry for Real and Even Signals	$x[n]$*is real and even*	a_k *is real and even*
Symmetry for Real and Odd Signals	$x[n]$ *is real and odd*	a_k *is purely inmaginary and odd*
Even-Odd Decomposition for Real Signals	$x_r[n] = Ev\{x[n]\}$ $[x(t) real]$ $x_o[n] = Od\{x[n]\}$ $[x(t) real]$	$Re\{a_k\}$ $jIm\{a_k\}$
Parseval's Relation for Periodic Signals	$\frac{1}{N} \sum_{n=<N>} \lvert x[n] \rvert^2 = \sum_{k=<N>} \lvert a_k \rvert^2$	

CONTINUOUS TIME FOURIER TRANSFORM

INTRODUCTION

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

PURPOSE

- Non-periodic signals can be represented with the help of Fourier Transform
- Fourier transform provides efficient reversible link between frequency domain and time domain representation of signals.
- For non-periodic signals, as time period tends to infinity and fundamental frequency tends to zero, spacing between spectral components becomes infinitesimal and spectrum appears to be continuous.

FOURIER SERIES FROM FOURIER TRANSFORM

Suppose that we are given a signal x(t) that is aperiodic. As a concrete example, suppose that x(t) is a square pulse, with x(t) = 1 if $-T_1 \le t \le T_1$, and zero elsewhere. Clearly x(t) is not periodic.

Now define a new signal $\tilde{x}(t)$,which is a periodic extension of x(t) with period T. In other words, $\tilde{x}(t)$ is obtained by repeating x(t), where each copy is shifted T units in time. This $\tilde{x}(t)$ has a Fourier series representation, which we found in the last section to be

$$a_0 = \frac{2T_1}{T}, \quad a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T},$$

Now recall that the Fourier series coefficients are calculated as follows:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt.$$

However, we note that x(t) = $\tilde{x}(t)$ in the interval of integration, and thus

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt.$$

Furthermore, since x(t) is zero for all t outside the interval of integration, we can expand the limits of the integral to obtain

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt.$$

Let us define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

This is called the Fourier transform of the signal x(t), and the Fourier series coefficients can be viewed as samples of the Fourier transform, scaled by T_1 , i.e.,

$$a_k = \frac{1}{T}X(jk\omega_0), k \in \mathbb{Z}.$$

Now consider the fact that

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}$$

Since $\omega_0 = \frac{2\pi}{T}$, this becomes

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

Now consider what happens as the period T gets bigger. In this case, $\tilde{x}(t)$ approaches x(t), and so the above expression becomes a representation of x(t). As $T \to \infty$, we have $\omega_0 \to 0$. Since each term in the summand can be viewed as the area of the rectangle whose height is $X(jk\omega_0)e^{-jk\omega_0 t}$ and whose base goes from $k\omega_0$ to $(k+1)\omega_0$, we see that as $\omega_0 \to 0$, the sum on the right-hand side approaches the area underneath the curve $X(j\omega)e^{-j\omega t}$ (where t is held fixed). Thus, as $T \to \infty$, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Thus, we have the following

Given a continuous-time signal x(t), the **Fourier Transform** of the signal is given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The **Inverse Fourier Transfo**rm of the signal is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

The Fourier transform $X(j\omega)$ is also called the spectrum of the signal, as it represents the contribution of the complex exponential of frequency ω to the signal x(t).

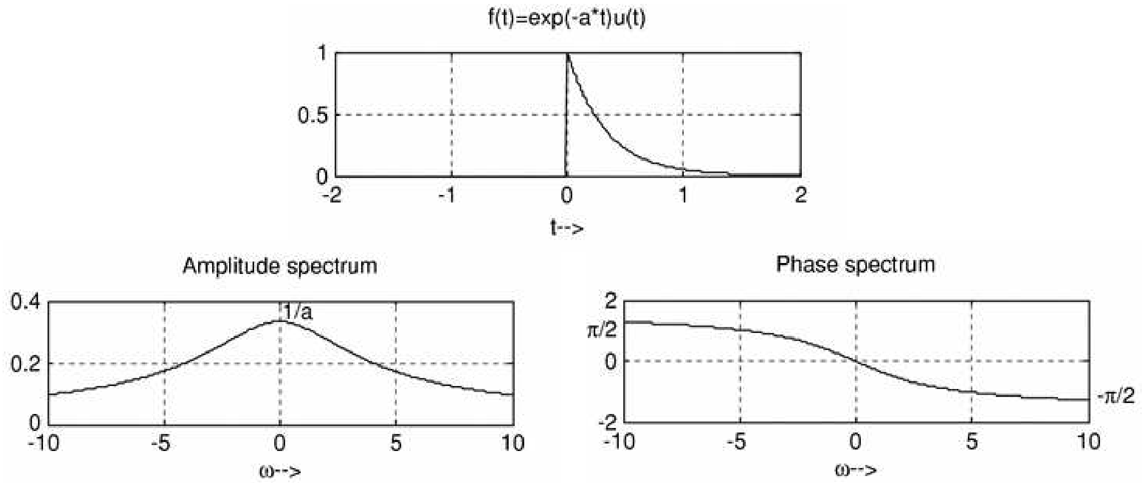
Example 1

Consider the signal $x(t) = e^{-at}u(t)$, a>0. Evaluate the Fourier transform of this signal.

$$\begin{aligned} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_{0}^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

To visualize $X(j\omega)$, we plot its magnitude and phase on separate plots (since $X(j\omega)$ is complex-valued in general). We have

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



Example 2

Find the Fourier transform of the signal x(t) = δ(t)

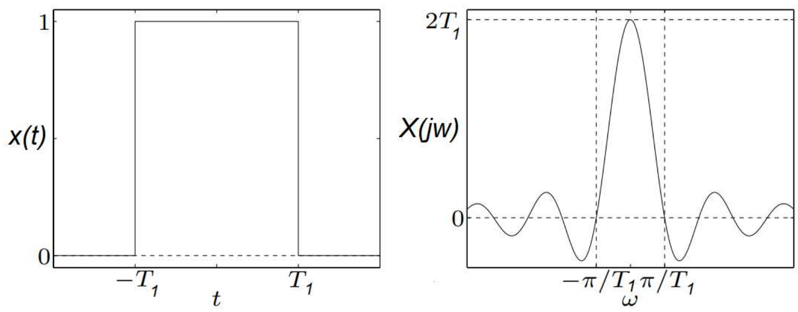
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

In other words, the spectrum of the impulse function has an equal contribution at all frequencies.

Example 3

Find the Fourier transform of the signal x(t) which is equal to 1 for $-T_1 \le t \le T_1$ and zero elsewhere.

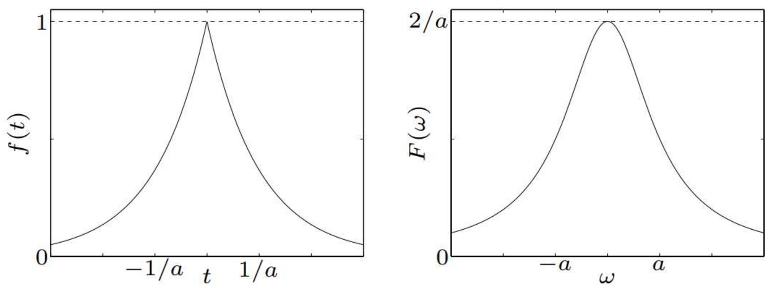
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) = \frac{2\sin(\omega T_1)}{\omega}$$



Example 4

Find the Fourier transform of $f(t) = e^{-a|t|}$, *where* $a > 0$

$$\begin{aligned} F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$



Example 5

Find the inverse Fourier transform of

$$X(j\omega) = \begin{cases} 1, & |\omega| \le W \\ 0 & |\omega| > W \end{cases}$$

$$\begin{aligned} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \frac{1}{jt}\, e^{j\omega t} \Big|_{-W}^{W} \\ &= \frac{\sin(Wt)}{\pi t}. \end{aligned}$$

The previous two examples showed the following. When x(t) is a square pulse, then $X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$ and when $X(j\omega)$ is a square pulse, $x(t) = \frac{\sin Wt}{\pi t}$.

EXISTENCE OF FOURIER TRANSFORM

There are a set of sufficient conditions called Dirichlet conditions under which a continuous-time signal x(t) is guaranteed to have a Fourier transform:

1. x(t) is absolutely integrable: $\int_{-\infty}^{\infty}|x(t)|dt < \infty$
2. x(t) has a finite number of maxima and minima in any finite interval.
3. x(t) has a finite number of discontinuities in any finite interval, and each of these discontinuities is finite.

PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

Suppose x(t) is a time-domain signal, and $X(j\omega)$ is its Fourier transform. We then say

$$X(j\omega) = \mathcal{F}\{x(t)\}$$
$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

We can also use the notation

$$x(t) \overset{\mathrm{F}}{\longleftrightarrow} X(j\omega)$$

to indicate that x(t) and $X(j\omega)$ are Fourier transform pairs.

1. Linearity
   If $F[x(t)] = X(j\omega)$ *and* $F[y(t)] = Y(j\omega)$, then for any constant a and b
   $$F[ax(t) + by(t)] = aX(j\omega) + bY(j\omega)$$
   Meaning: The Fourier transform of linear combination of signals is equal to their linear combination of their Fourier transforms.
2. Time Shifting
   If $F[x(t)] = X(j\omega)$, then for any constant t_0,
   $$F[x(t - t_0)] = e^{-j\omega t_0}X(j\omega)$$
   Meaning: A shift of t_0 in time domain is equivalent to introducing a phase shift of $-\omega t_0$. Amplitude remains the same
3. Frequency shifting
   If $F[x(t)] = X(j\omega)$, then
   $$e^{j\beta t}x(t) = X(w - \beta)$$
   Meaning: By shifting the frequency by β in frequency domain is equivalent to multiplying the time domain by $e^{j\beta t}$.
4. Time scaling
   If $F[x(t)] = X(j\omega)$, then for any constant a, then
   $$F[x(at)] = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

   Meaning: Compression of a signal in time domain is equivalent to expansion in frequency domain and vice-versa.

5. Time differentiation

   If $F[x(t)] = X(j\omega)$,then

   $$\frac{dF[x(t)]}{dt} = j\omega X(j\omega)$$

   Meaning: Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain.

6. Frequency differentiation

   If $F[x(t)] = X(j\omega)$,then

   $$F[-jt\, x(t)] = \frac{dX(j\omega)}{d\omega}$$

   Meaning: Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by the complex number -jt.

7. Integration

   If $F[x(t)] = X(j\omega)$,then

   $$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

   Meaning: Integration in time domain represents smoothing in frequency domain.

8. Convolution

   If $F[x(t)] = X(j\omega)\ and\ F[h(t)] = H(j\omega)$, then

   $$F[x(t) * h(t)] = X(j\omega)H(j\omega)$$

   Meaning: Fourier transform of convolution of two signals in time domain is equal to the product of individual Fourier transforms

9. Modulation

   If $F[x(t)] = X(j\omega)\ and\ F[z(t)] = Z(j\omega)$, then

   $$F[x(t)z(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)Z(\omega - \xi)d\xi = \frac{1}{2\pi}[X(j\omega) * Z(j\omega)]$$

   Meaning: Fourier transform of product of two signals is equal to the convolution of their individual Fourier transforms.

10. Duality

    If $F[x(t)] = X(j\omega)$, then

    $$F[X(t)] = 2\pi\, x(-j\omega)$$

11. Symmetry

    Let x(t) be a real signal with $x_e(t)$ and $x_o(t)$ as its even and odd part and $X(j\omega) = X_R(j\omega) + jX_I(j\omega)$, then

    $$F[x_e(t)] = X_R(j\omega) \text{ and } F[x_o(t)] = jX_I(j\omega)$$

Property	Aperiodic signal	Fourier transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

DISCRETE TIME FOURIER TRANSFORM

INTRODUCTION

We saw that the Fourier transform for continuous-time aperiodic signals can be obtained by taking the Fourier series of an appropriately defined periodic signal (and letting the period go to infinity); we will follow an identical argument for discrete-time aperiodic signals. The differences between the continuous-time and discrete-time Fourier series will be reflected as differences between the continuous-time and discrete-time Fourier transforms as well.

DISCRETE TIME FOURIER TRANSFORM FROM DISCRETE TIME FOURIER SERIES

Consider a general signal x[n] which is nonzero on some interval $-N_1 \leq n \leq N_2$ and zero elsewhere. We create a periodic extension $\tilde{x}[n]$ of this signal with period N (where N is large enough so that there is no overlap). As $N \to \infty$, $\tilde{x}[n]$ becomes equal to x[n] for each finite value of n.

Since $\tilde{x}[n]$ is periodic, it has a discrete-time Fourier series representation given by

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

Where $\omega_0 = \frac{2\pi}{N}$. The Fourier series coefficients are given by

$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} \tilde{x}[n] e^{-j\omega_0 kn}$$

where n_0 is any integer. Suppose we choose n_0 so that the interval $[-N_1, N_2]$ is contained in $[n_0, n_0+N-1]$. Then since $\tilde{x}[n]$ = x[n] in this interval, we have

$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\omega_0 kn} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 kn}$$

The **discrete time Fourier transform** is defined as

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

From this, we see that $a_k = \frac{1}{N} X(e^{jk\omega_0})$, i.e., the discrete-time Fourier series coefficients are obtained by sampling the discrete-time Fourier transform at periodic intervals of ω_0. Also note that $X(e^{j\omega_0})$ is periodic in ω with period 2π(since $e^{-j\omega n}$ is 2π-periodic).

Using the Fourier series representation of $\tilde{x}[n]$, we now have

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0.$$

Once again, we see that each term in the summand represents the area of a rectangle of width ω_0 obtained from the curve $X(e^{j\omega})e^{j\omega}$. As $N \to \infty$, we have $\omega_0 \to 0$,. In this case, the sum of the areas of the rectangles approaches the integral of the curve $X(e^{j\omega})e^{j\omega n}$, and since the sum was over only N samples of the function, the integral is only over one interval of length 2π. Since $\tilde{x}[n]$ approaches x[n] as $N \to \infty$, we have

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

This is the **inverse discrete-time Fourier transform**, or the synthesisequation.

The main differences between the discrete-time and continuous-time Fourier transforms are the following.

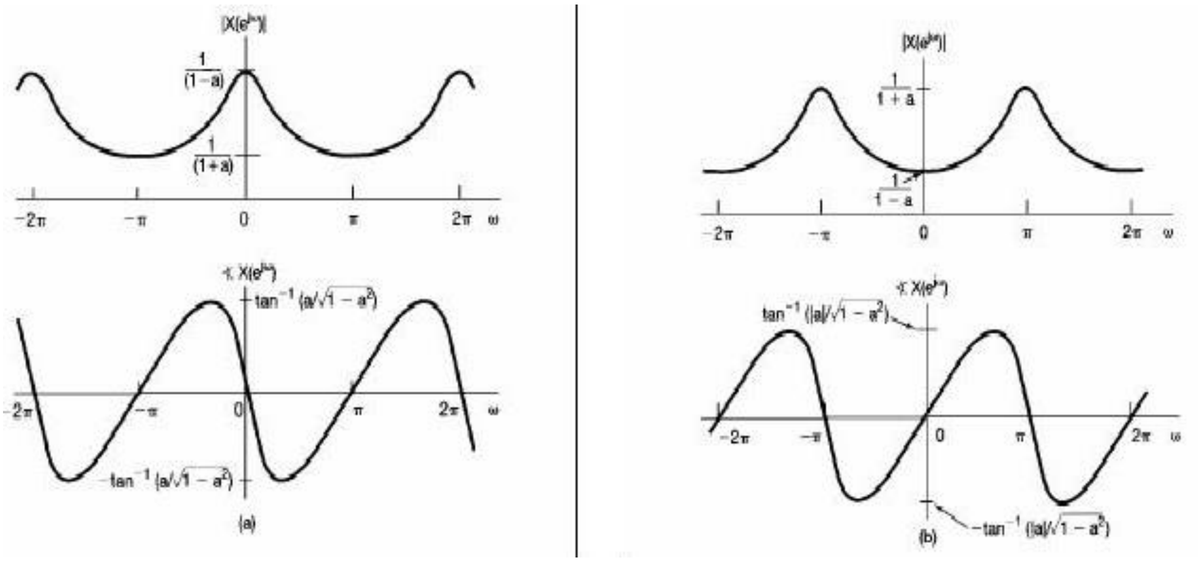
1) The discrete-time Fourier transform $X(e^{j\omega})$ is periodic in ω with period 2π, whereas the continuous-time Fourier transform is not necessarily periodic.
2) The synthesis equation for the discrete-time Fourier transform only involves an integral over an interval of length 2π, whereas the one for the continuous-time Fourier transform is over the entire ω axis. Both of these are due to the fact that $e^{j\omega n}$ is 2π-periodic in ω, whereas the continuous-time complex exponential is not.
3)

	Time Domain	Frequency Domain	Time Domain	Frequency domain
Fourier Series FS	$x(t)$ ←FS→ $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ Continuous/Periodic	a_k $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}dt$ Aperiodic/Discrete	$x[n]$ ←DTFS→ $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$ Discrete/Periodic	a_k Duality $a_k = \frac{1}{N}\sum_{k=<N>} x[n]e^{-jk\omega_0 n}$ Periodic/Discrete
Fourier Transform FT	$x(t)$ ←FT→ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}d\omega$ Continuous/aperiodic	$X(j\omega)$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}dt$ Aperiodic/Continuous Duality	$x[n]$ ←DTFT→ $x[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}d\omega$ Discrete/aperiodic	$X(e^{j\omega})$ $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$ periodic/Continuous

Example 1

Find the DTFT of *x[n]=aⁿ u[n]*

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$
$$= \frac{1}{1 - ae^{-j\omega}}.$$

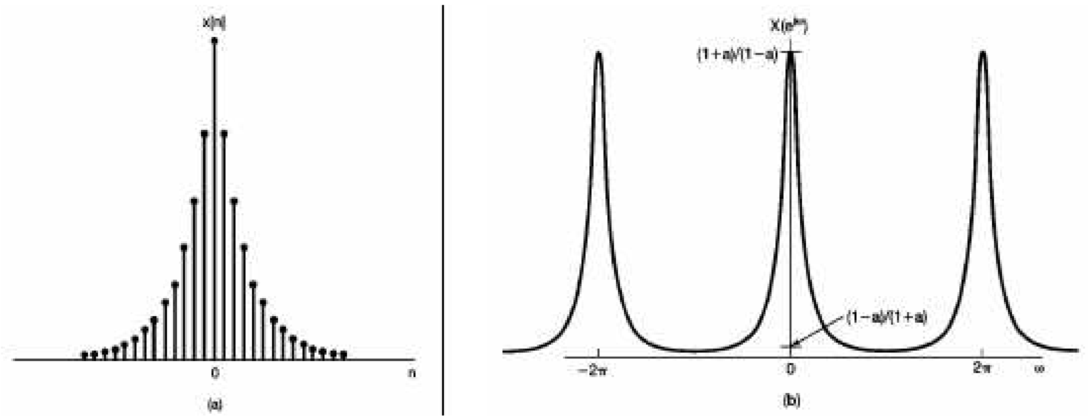


If we plot the magnitude of $X(e^{j\omega})$, we see an illustration of the "high" versus "low" frequency effect. Specifically, if a > 0 then the signal x[n] does not have any oscillations and $|X(e^{j\omega})|$ has its highest magnitude around even multiples of π. However, if a < 0, then the signal x[n] oscillates between positive and negative values at each time-step; this "high-frequency" behaviour is captured by the fact that $|X(e^{j\omega})|$ has its largest magnitude near odd multiples of π.

Example 2

Find the DTFT of $x[n]=a^{|n|}\ u[n]$

$$
\begin{aligned}
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n}e^{-j\omega n} \\
&= \sum_{n=1}^{\infty} a^{n}e^{j\omega n} + \sum_{n=0}^{\infty} a^{n}e^{-j\omega n} \\
&= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} \\
&= \frac{1-a^2}{1-2a\cos(\omega)+a^2}.
\end{aligned}
$$



PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Let x[n] and y[n] be two signals, then their DTFT is denoted by $X(e^{j\omega})$ and $Y(e^{j\omega})$.

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

The notation is used to say that left hand side is the signal x[n] whose DTFT is $X(e^{j\omega})$ is given at right hand side.

1. Periodicity
   The discrete-time Fourier transform is always periodic in ω with period 2π.
   $$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

2. Linearity of DTFT
   If $x_1[n] \overset{F}{\longleftrightarrow} X_1(e^{j\omega})$, and $x_2[n] \overset{F}{\longleftrightarrow} X_2(e^{j\omega})$, then
   $$ax_1[n] + bx_2[n] \overset{F}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

3. Time shifting and frequency shifting
   If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$,

   then
   $$x[n-n_0] \overset{F}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$
   and
   $$e^{j\omega_0 n} x[n] \overset{F}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

4. Differencing and accumulation
   If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$,

   then
   $$x[n] - x[n-1] \overset{F}{\longleftrightarrow} (1 - e^{-j\omega}) X(e^{j\omega}).$$

   For signal
   $$y[n] = \sum_{m=-\infty}^{n} x[m],$$

   its Fourier transform is given as
   $$\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi k).$$

5. Time reversal

   If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

   then

   $$\boxed{x[-n] \xleftrightarrow{F} X(-e^{j\omega})}.$$

6. Time expansion

   Let us define a signal with k a positive integer,

   $$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}.$$

   Its Fourier transform is given by

   $$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk} = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

   That is,

   $$\boxed{x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})}.$$

7. Differentiation in frequency

   If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

   Differentiate both sides of the analysis equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

   $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

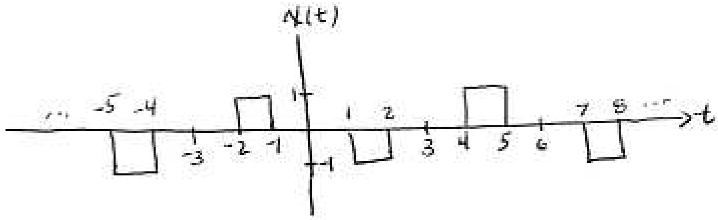
8. Convolution

   If $F[x(n)] = X(e^{j\omega})$ and $F[h(n)] = H(e^{j\omega})$, then

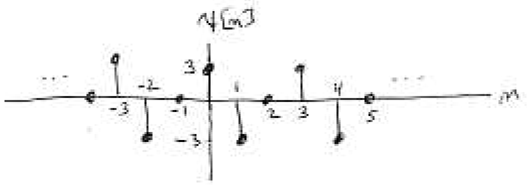
   $$F[x(n) * h(n)] = X(e^{j\omega})H(e^{j\omega})$$

UNSOLVED PROBLEMS

1. Compute the complex-form Fourier series coefficients and sketch the magnitude and phase spectra for
   a. the signal x(t) that has fundamental period T_0 =1, with x(t)=e^{-t} 0≤ t ≤1.
   b. the signal shown below



2. Compute the discrete-time Fourier series coefficients for the signals below and sketch the magnitude and phase spectra.



   a.
   b. $x[n] = \sum_{k=-\infty}^{\infty} \delta(n - 4k - 1)$
3. From the basic definition, compute the Fourier transforms of the signals.
   a. $x[n] = e^{-(t-2)}u(t-3)$
   b. $x[n] = e^{-|t+1|}$
4. Use the linearity of the DTFT to determine the DTFT of the following sum of two right-sided exponential signals: x[n] = $(0.8)^n u[n] + 2(-0.5)^n u[n]$.
5. Find the DTFT of $x[n] = 7u[n-1] - 7u[n-9]$
6. Determine the inverse DTFT of $Y(e^{j\omega}) = 6\cos(3\omega)$

Revision 2021

Semester 5

SIGNALS & SYSTEMS

MODULE 4 NOTES

OUTCOMES & CONTENTS

Module Outcomes	Description	Duration (Hours)	Cognitive Level
CO4	**Apply Laplace transform to demonstrate the concepts of signals andsystems**		
M4.01	Interpret the frequency domain parameters ofa signal using Laplace transform	5	Understan ding
M4.02	Illustrate the region of convergence	2	Understan ding
M4.03	Outline Properties of Laplace transform	4	Understan ding
M4.04	Apply Inverse Laplace transform to Signals	5	Applying

Contents:

Need of Laplace transform

Region of Convergence (ROC)

Advantages and limitation of Laplace transform

Laplace transform of some commonly used signals - impulse, step, ramp, parabolic, exponential, sine and cosine functions

Properties of Laplace transform: Linearity, time shifting, time scaling, time reversal, transform of derivatives and integrals, initial value theorem, final value theorem.

Inverse Laplace transform: simple problems (no derivation required)

LAPLACE TRANSFORM

The Laplace transform is named after Pierre Simon De Laplace, a famous French mathematician (1749-1827) who formulated a transformation that can **convert one signal into another using a set of laws or equations.**

The Laplace transformation is the most effective method for converting differential equations to algebraic equations. In electronics engineering, the Laplace transformation is very important to solve problems related to **signal and system, digital signal processing,** and control system. In studying the dynamic control system, the characteristics of the **Laplace transform and the inverse Laplace transformation are both employed.**

A piecewise continuous function is a function that has a finite number of breaks, and this consistency remains till the function reaches infinity.

First Let f(t) be the function of t, time for all t ≥ 0

- Then the Laplace transform of f(t), F(s) can be defined as

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

provided that the integral exists, where the Laplace Operator, s = σ + jω; will be real or complex with j = √(-1).

The Laplace transform will change the differential equation into an easy-to-solve algebraic function. This transform converts any signal into the frequency domain 's', where the complexity of the problem reduces. **Whenever you encounter any function written inside the capital letter L, instantly identify that it is the Laplace transform of a function.** It can also be represented as the capital letter of the function related to frequency 's'. For instance, F (s), A (s), G (s), X (s), etc.

We say that F(s) is the Laplace Transform of f(t),

$$\mathcal{L}\left(f(t)\right) = F(s)$$

or that f(t) is the inverse Laplace Transform of F(s),

$$\mathcal{L}^{-1}\left(F(s)\right) = f(t)$$

or that f(t) and F(s) are a Laplace Transform pair,

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

Laplace transform is formulated as the integration of the product of the function with e^{-st}, with limits ranging from 0 to infinity.

There are certain steps which need to be followed in order to do a Laplace transform of a time function. In order to transform a given function of time f(t) into its corresponding Laplace transform, we have to follow the following steps:

- First multiply f(t) by e^{-st}, s being a complex number (s = σ + j ω).
- Integrate this product w.r.t time with limits as zero and infinity. This integration results in Laplace transformation of f(t), which is denoted by F(s).

$$Laplace\ transform\ of\ f(t) = \pounds\left[f(t)\right] = F(s) = \int_0^{\infty} f(t)\cdot e^{-st}dt$$
$$when\ t \geq 0$$

The time function f(t) is obtained back from the Laplace transform by a process called inverse Laplace transformation and denoted by $\pounds^{-1}$

$$Inverse\ Laplace\ transform\ of\ F(s) = \pounds^{-1}\left[F(s)\right] = \pounds^{-1}\left[\pounds f(t)\right] = f(t)$$

The Laplace transform is performed on a number of functions, which are – impulse, unit impulse, step, unit step, shifted unit step, ramp, exponential decay, sine, cosine, hyperbolic sine, hyperbolic cosine, natural logarithm, Bessel function. But the greatest advantage of applying the Laplace transform is solving higher order differential equations easily by converting into algebraic equations.

Laplace transform is divided into two types, namely

- **one-sided Laplace transformation**
- **two-sided Laplace transformation.**

One-sided Laplace transformation: The Laplace transformation with **limits 0 to infinity** is known as one-sided. This is also known as unilateral Laplace transformation.

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Two-sided Laplace transformation: The Laplace transformation with limits ranging from -infinity to +infinity is considered as two-sided. This transformation is also known as **bilateral Laplace** transformation.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

APPLICATIONS OF LAPLACE TRANSFORM

- Analysis of electrical and **electronic circuits.**
- Breaking down complex differential equations into simpler polynomial forms.
- Laplace transform gives information about steady as well as transient states.
- In machine learning, the Laplace transform is used for making predictions and making analysis in data mining.

PHYSICAL SIGNIFICANCE OF LAPLACE TRANSFORM

Laplace transform has no physical significance except that it transforms the time domain signal to a complex frequency domain. It is useful to simply the mathematical computations and it can be used for the easy analysis of signals and systems.

- System modelling: Laplace Transform is used to simplify calculations in system modelling, where large number of differential equations are used
- In the telecommunications industry, it is utilized to deliver signals to both sides of the medium.
- It is also utilized for a variety of technical jobs, including electrical circuit analysis, digital signal processing, system modelling, and more.

APPLICATION OF LAPLACE TRANSFORM IN SIGNAL PROCESSING

Laplace transforms are frequently opted for signal processing. Along with the Fourier transform, the **Laplace transform** is used to study signals in the frequency domain. When there are small frequencies in the signal in the frequency domain then one can expect the signal to be smooth in the time domain. Filtering of a signal is usually done in the frequency domain for which Laplace acts as an important tool for converting a signal from time domain to frequency domain.

APPLICATION OF LAPLACE TRANSFORM IN CONTROL SYSTEMS

Control systems are usually designed to control the behaviour of other devices. Example of **control systems** can range from a simple home heating controller to an industrial control system regulates the behaviour of machinery.

Generally, control engineers use differential equations to describe the behaviour of various closed loop functional blocks. Laplace transform is used here for solving these equations without the loss of crucial variable information

Transforms are not necessary to deal with the signal and system, but

- They make signal formation easy and convenient.
- These are the best ways to deal with the system.
- Computation and analysis have become interesting and convenient.

REGION OF CONVERGENCE

Region of Convergence (ROC) is defined as the set of points in s-plane for which the Laplace transform of a function x(t) converges. In other words, the range of Re(s) (i.e. σ) for which the function X(s)converges is called the region of convergence.

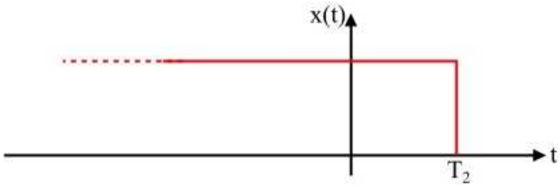
ROC of Right-Sided Signals

A signal x(t) is said to be a right-sided signal if the signal x(t)= 0 for t < T_1 for some finite time T_1

For a right-sided signal x(t), the ROC of the Laplace transform X(s) is Re(s)>σ_1, where σ_1 is a constant. Thus, the ROC of the Laplace transform of the right-sided signal is to the right of the line $\sigma=\sigma_1$. A causal signal is an example of a right-sided signal.

ROC of Left-Sided Signals

A signal x(t) is said to be a left-sided signal if the signal x(t) = 0 for *t* > T_2 for some finite time T_2.

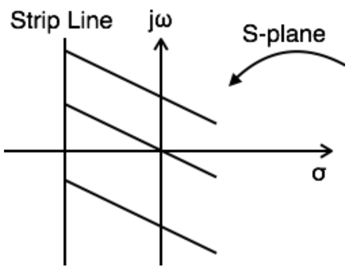


For a left-sided signal x(t), the ROC of the Laplace transform X(s) is Re(s)<σ2, where σ_2 is a constant. Therefore, the ROC of the Laplace transform of a left-sided signal is to the left of the line $\sigma = \sigma_2$. An anti-causal signal is an example of a left-sided signal.

The range variation of σ for which the Laplace transform converges is called region of convergence.

PROPERTIES OF ROC OF LAPLACE TRANSFORM

- ROC contains strip lines parallel to jω axis in s-plane.



- If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x(t) is a right sided sequence then ROC: Re{s} > σ_o.
- If x(t) is a left sided sequence then ROC: Re{s} < σ_o.
- If x(t) is a two-sided sequence then ROC is the combination of two regions.

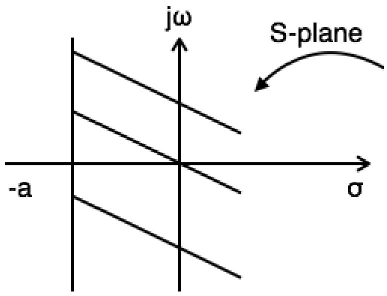
ROC can be explained by making use of examples given below:

Example 1:

Find the Laplace transform and ROC of $x(t) = e^{-at}u(t)$

$L[x(t)] = L[e^{-at}u(t)] = \frac{1}{s+a}$

ROC: Re{s} >> −a

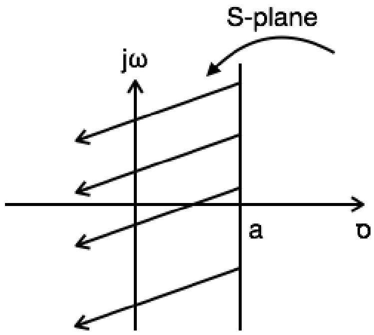


Example 2:

Find the Laplace transform and ROC of $x(t) = e^{at}u(-t)$.

$L[x(t)] = L[e^{at}u(-t)] = \frac{1}{s-a}$

ROC: Re{s} < a



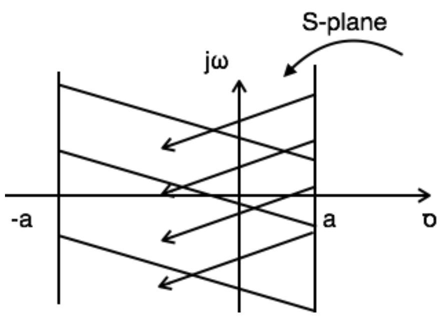
Example 3:

Find the Laplace transform and ROC of $x(t) = e^{-at}u(t) + e^{at}u(-t)$

$$L[x(t)] = L[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{s+a} + \frac{1}{s-a}$$

For 1/s+a , Re{s}> −a

For -1/s−a, Re{s}< a

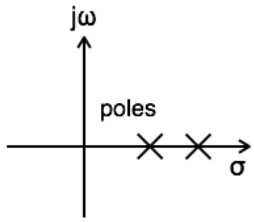


Referring to the above diagram, combination region lies from –a to a. Hence,

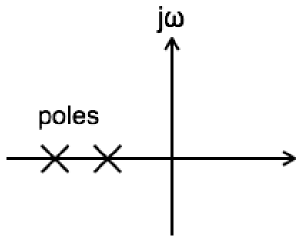
ROC: −a<Re{s}<a

CAUSALITY AND STABILITY

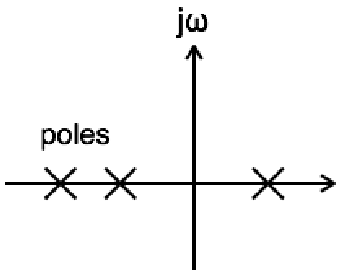
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



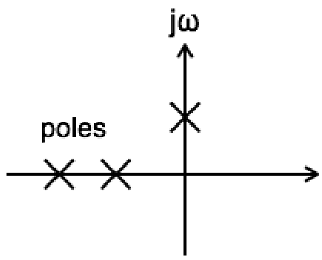
- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transfer function lies on the jω axis of s-plane.



ROC OF BASIC FUNCTIONS

f(t)	F(s)	ROC
u(t)	1/s	Re{s} > 0
t.u(t)	$1/s^2$	Re{s} > 0
t^n.u(t)	$n! / s^{n+1}$	Re{s} > 0
$e^{at}u(t)$	1/s-a	Re{s} > a
e^{-at} u(t)	1/s+a	Re{s} > -a
$e^{at}u(-t)$	−1/s-a	Re{s} < a
e^{-at} u(−t)	−1/s+a	Re{s} < -a
t $e^{at}u(t)$	$1/(s-a)^2$	Re{s} > a
$t^n e^{at}u(t)$	$n! / (s-a)^{n+1}$	Re{s} > a
$te^{-at}u(t)$	$1/(s+a)^2$	Re{s} > -a
$t^n e^{-at}$ u(t)	$n! /(s+a)^{n+1}$	Re{s} > -a
t $e^{at}u(-t)$	$-1/(s-a)^2$	Re{s} < a
$t^n e^{at}u(-t)$	$-n! /(s-a)^{n+1}$	Re{s} < a
t $e^{-at}u(-t)$	$-1/(s+a)^2$	Re{s} < -a
$t^n e^{-at}u(-t)$	$n! /(s+a)^{n+1}$	Re{s} < -a
e^{-at}cosbt	$(s+a)/(s+a)^2+b^2$	
e^{-at}sinbt	$b/(s+a)^2+b^2$	

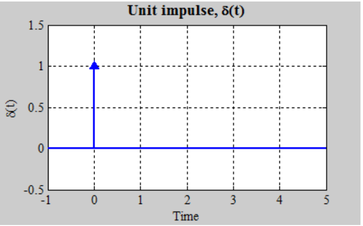
LAPLACE TRANSFORM OF SOME COMMONLY USED SIGNALS

1.The Unit Impulse

The impulse function is everywhere but at t=0, where it is infinitely large. The area of the impulse function is one. The impulse function is drawn as an arrow whose height is equal to its area.

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



To find the Laplace Transform, we apply the definition

$$\Delta(s) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt$$

Now we apply the shifting property of the impulse. Since the impulse is 0 everywhere but t=0, we can change the upper limit of the integral to 0^+.

$$\Delta(s) = \int_{0^-}^{0^+} \delta(t)e^{-st}dt$$

Since e^{-st} is continuous at t=0, that is the same as saying it is constant from $t=0^-$ to $t=0^+$. So, we can replace e^{-st} by its value evaluated at t=0.

$$\left. e^{-st} \right|_{t=0} = e^{-s\cdot 0} = 1$$

$$\Delta(s) = \int_{0^-}^{0^+} \delta(t) \cdot 1 \cdot dt = 1$$

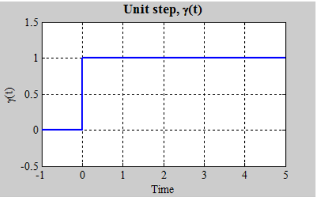
$$\delta(t) \xleftrightarrow{\mathcal{L}} 1$$

So, the Laplace Transform of the unit impulse is just one. Therefore, the impulse function, which is difficult to handle in the time domain, becomes easy to handle in the Laplace domain. It will turn out that the unit impulse will be important to much of what we do.

2. The Unit Step Function

The unit step function is defined as

$$\gamma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



To find the Laplace Transform, we apply the definition.

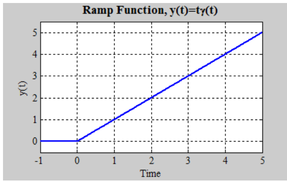
$$\begin{aligned}\Gamma(s) &= \int_{0^-}^{\infty} \gamma(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} 1 \cdot e^{-st}dt = -\frac{1}{s}\left[e^{-st}\right]_{0^-}^{\infty} \\ &= -\frac{1}{s}\left[0-1\right] \\ &= \frac{1}{s}\end{aligned}$$

so

$$\gamma(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} = \Gamma(s)$$

3. The Ramp

So far (with the exception of the impulse), all the functions have been closely related to the exponential. It is also possible to find the Laplace Transform of other functions. For example, the ramp function:



$y(t) = t, t > 0;$

$= 0$, elsewhere

$Y(S) = {}_0\int^{\infty} y(t).\ e^{-st}.dt.$

$= {}_0\int^{\infty} t.\ e^{-st}.dt.$

$= [t.e^{-st}/-s]_0^{\infty} - {}_0\int^{\infty} e^{-st}.(1/-s).dt.$

$= 1/s^2$

Integration by parts is useful at this point

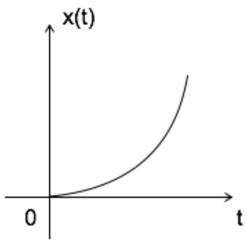
Therefore **$L[t] = 1/s^2$**

4. The Parabolic

A unit parabolic function is defined as

$y(t) = t^2/2$; for $t > 0$

$= 0$; elsewhere



$Y(S) = {}_0\int^{\infty} y(t).\ e^{-st}.dt.$

$= {}_0\int^{\infty} t^2/2.\ e^{-st}.dt.$

$= [t^2/2.e^{-st}/-s]_0^{\infty} - {}_0\int^{\infty} t.e^{-st}.(1/-s).dt.$

$=1/s[{}_0\int^{\infty} t.e^{-st}dt.]$

$=1/s\ .1/s^2$

$=1/s^3$

Therefore **$L[t^2/2] = 1/s^3$**

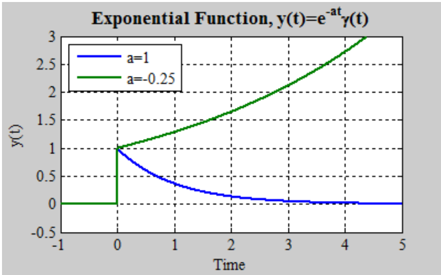
5. The Exponential

Consider the causal (i.e., defined only for t>0) exponential:

$$y(t) = \begin{cases} 0, & t < 0 \\ e^{-at}, & t \geq 0 \end{cases}$$

or

$$y(t) = e^{-at}\gamma(t)$$



To find the Laplace Transform, we apply the definition

$$Y(s) = \int_{0^-}^{\infty} y(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-at}\gamma(t)e^{-st}dt$$

Since γ(t) is equal to one for all positive t, we can remove it from the integral

$$Y(s) = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = \int_{0^-}^{\infty} e^{-(s+a)t}dt$$

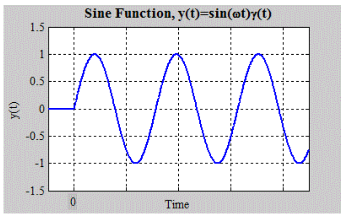
$$= \int_{0^-}^{\infty} 1 \cdot e^{-(s+a)t}dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_{0^-}^{\infty} = -\frac{1}{s+a}[0-1]$$

$$= \frac{1}{s+a}$$

$$e^{-at} \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

6. The Sine

$$y(t) = \sin(\omega t)\gamma(t)$$



As before, start with the definition of the Laplace transform

$$Y(s) = \int_{0^-}^{\infty} \sin(\omega t)e^{-st}dt$$

Here it becomes useful to use Euler's identity for the sine

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

so

$$Y(s) = \int_{0^-}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st}dt = \frac{1}{2j}\int_{0^-}^{\infty} e^{j\omega t}e^{-st}dt - \frac{1}{2j}\int_{0^-}^{\infty} e^{-j\omega t}e^{-st}dt$$

But we've already done this integral (the exponential function, above)

$$Y(s) = \frac{1}{2j}\frac{1}{s-j\omega} - \frac{1}{2j}\frac{1}{s+j\omega}$$

Let's put this over a common denominator

$$Y(s) = \frac{1}{2j}\frac{1}{(s-j\omega)}\frac{(s+j\omega)}{(s+j\omega)} - \frac{1}{2j}\frac{1}{(s+j\omega)}\frac{(s-j\omega)}{(s-j\omega)}$$

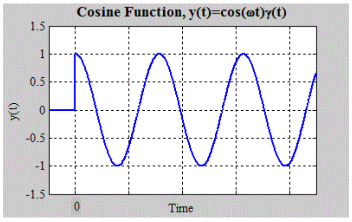
$$= \frac{1}{2j}\frac{(s+j\omega)-(s-j\omega)}{\left(s^2 - \cancel{sj\omega} + \cancel{sj\omega} - (j\omega)^2\right)} = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2}$$

$$\sin(\omega t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega}{s^2+\omega^2}$$

7.The Cosine

$$y(t) = \cos(\omega t)\gamma(t)$$



The cosine can be found in much the same way, but using Euler's identity for the cosine.

$$\cos(\omega t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega^2}$$

PROPERTIES OF LAPLACE TRANSFORM

1. Linearity

$$f_1(t) \xrightarrow{L.T.} F_1(s) \ \ with\ ROC = \ R_1$$

$$f_2(t) \xrightarrow{L.T.} F_2(s) \ \ with\ ROC = \ R_2$$

$$af_1(t) + \ bf_2(t) \xrightarrow{L.T.} aF_1(s) + \ bF_2(s){:}\ \ ROC = \ R_1 \cap R_2$$

2. Time Shifting

$$f(t) \xrightarrow{L.T.} F(s) \ \ with\ ROC = \ R$$

$$f(t - t_0) \xrightarrow{L.T.} e^{-st_0} F(s){:}\ ROC = R$$

3. Time Scaling:

$$f\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{L}} aF(as)$$

4. Shift in S-domain

$$f(t) \xrightarrow{L.T.} F(s) \ \ with\ ROC = \ R$$

$$e^{st_0} f(t)) \xrightarrow{L.T.} F(s - s_0){:}\ ROC = R + Re\{s_0\}$$

5. Time-reversal

$$f(t) \xrightarrow{L.T.} F(s) \ \ with\ ROC = \ R$$

$$f(-t) \xrightarrow{L.T.} F(-s) \ \ with\ ROC = \ -R$$

6. Differentiation in S-domain

$$f(t) \xrightarrow{L.T.} F(s)\ with\ ROC = R_1$$

$$tf(t) \xrightarrow{L.T.} \frac{-d}{ds} F(s){:}\ ROC = R$$

The differentiation property of the Laplace Transform. We will use the differentiation property widely. It is repeated below (for first, second and n^{th} order derivatives)

$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{L}} sF(s) - f(0^-)$$

$$\frac{d^2f(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\frac{d^nf(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^nF(s) - s^{(n-1)}f(0^-) - s^{(n-2)}\dot{f}(0^-) - \cdots - s\, f^{(n-2)}(0^-) - f^{(n-1)}(0^-)$$

7. Integration

The integration theorem states that

$$\int_{0^-}^{t} f(\lambda)d\lambda \xleftrightarrow{\mathcal{L}} \frac{F(s)}{s}$$

$$\mathcal{L}\left(\int_{0^-}^{t} f(\lambda)d\lambda\right) = \frac{1}{s}F(s)$$

8. Convolution in Time

$$\text{if } f(t) \xrightarrow{L.T.} F(s) \text{ with } ROC = R_1 \text{ and } h(t) \xrightarrow{L.T.} H(s) \text{with } ROC = R_2$$

$$f(t) * h(t) \xrightarrow{L.T.} F(s)H(s): ROC = R_1 \cap R_2$$

9. Initial Value Theorem

Initial value theorem is applied when in Laplace transform the degree of the numerator is less than the degree of the denominator

$$f(0) = \lim_{s\to\infty} sF(s)$$

10. Final Value Theorem:

If all the poles of sF(s) lie in the left half of the S-plane final value theorem is applied.

$$f(\infty) = \lim_{s\to 0} sF(s)$$

11. Multiplication by time:

$$tf(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{dF(s)}{ds}$$

12. Complex Shift:

$$f(t)e^{-at} \xleftrightarrow{\mathcal{L}} F(s+a)$$

Properties with proof (For reference)

1. Linearity

The linearity property of the Laplace Transform states:

$$a \cdot f(t) + b \cdot g(t) \xleftrightarrow{\mathcal{L}} a \cdot F(s) + b \cdot G(s)$$

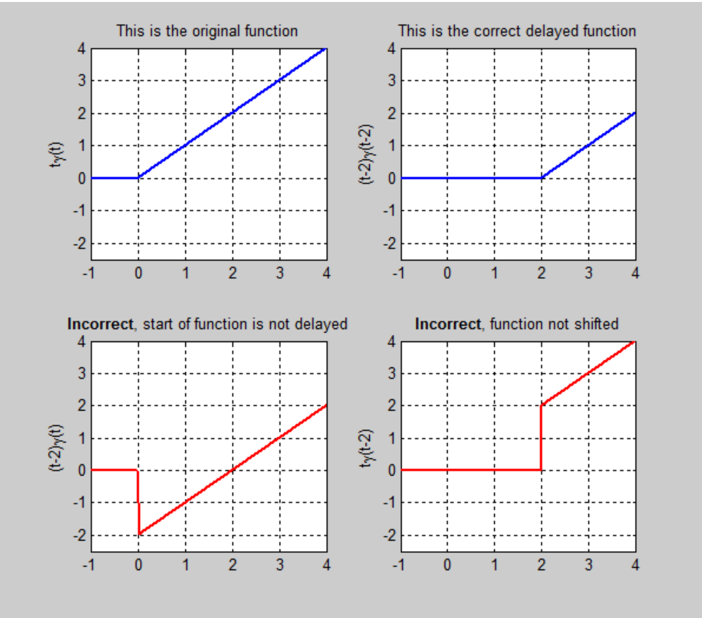
This is easily proven from the definition of the Laplace Transform

$$\begin{aligned}\mathcal{L}(a \cdot f(t) + b \cdot g(t)) &= \int_{0^-}^{\infty} (a \cdot f(t) + b \cdot g(t)) e^{-st} dt \\ &= a\int_{0^-}^{\infty} f(t) e^{-st} dt + b\int_{0^-}^{\infty} g(t) e^{-st} dt \\ &= a \cdot F(s) + b \cdot G(s)\end{aligned}$$

2. Time Delay

The time delay property is not much harder to prove, but there are some subtleties involved in understanding how to apply it. We'll start with the statement of the property, followed by the proof, and then followed by some examples. The time shift property states

$$f(t-a) \cdot \gamma(t-a) \xleftrightarrow{\mathcal{L}} e^{-as}F(s)$$



The correct one is exactly like the original function but shifted.

Important: To apply the time delay property you must multiply a delayed version of your function by a delayed step. If the original function is $g(t)\cdot\gamma(t)$, then the shifted function is $g(t-t_d)\cdot\gamma(t-t_d)$ where t_d is the time delay.

3. First Derivative

The first derivative property of the Laplace Transform states

$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{L}} sF(s) - f(0^-)$$

To prove this we start with the definition of the Laplace Transform and integrate by parts

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$\int_a^b u \cdot dv = u \cdot v\Big|_a^b - \int_a^b v \cdot du$$

$$du = -s \cdot e^{-st} dt \qquad u = e^{-st}$$

$$v = f(t) \qquad dv = \frac{df(t)}{dt} dt$$

$$\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[e^{-st} \cdot f(t)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t) \cdot (-s) \cdot e^{-st} dt$$

$$= \left[e^{-st} \cdot f(\infty) - e^{-0^- t} \cdot f(0^-)\right] + s\int_{0^-}^{\infty} f(t) \cdot e^{-st} dt$$

The first term in the brackets goes to zero (as long as f(t) doesn't grow faster than an exponential which was a condition for existence of the transform). In the next term, the exponential goes to one. The last term is simply the definition of the Laplace Transform multiplied by s. So the theorem is proved.

$$\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[\cancel{e^{-st} f(\infty)} - \cancel{e^{-0^- t}} \cdot f(0^-)\right] + sF(s) = sF(s) - f(0^-)$$

There are two significant things to note about this property:

- We have taken a derivative in the time domain, and turned it into an algebraic equation in the Laplace domain. This means that we can take differential equations in time, and turn them into algebraic equations in the Laplace domain. We can solve the algebraic equations, and then convert back into the time domain (this is called the Inverse Laplace Transform, and is described later).
- The initial conditions are taken at $t=0^-$. This means that we only need to know the initial conditions before our input starts. This is often much easier than finding them at $t=0^+$.

Second Derivative

Similarly for the second derivative we can show:

$$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2 F(s) - sf(0^-) - \dot{f}(0^-)$$

where

$$\dot{f}(0^-) = \left.\frac{df(t)}{dt}\right|_{0^-}$$

N^{th} order Derivative

For the n^{th} derivative:

$$\frac{d^n f(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^n F(s) - s^{(n-1)} f(0^-) - s^{(n-2)} \dot{f}(0^-) - \cdots - s\, f^{(n-2)}(0^-) - f^{(n-1)}(0^-)$$

or

$$\frac{d^n f(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^n F(s) - \sum_{i=1}^{n} s^{(n-i)} \overset{(i-1)}{f}(0^-)$$

where

$$\overset{(n)}{f}(0^-) = \left.\frac{d^n f(t)}{dt^n}\right|_{0^-}$$

Key Concept: The differentiation property of the Laplace Transform

We will use the differentiation property widely. It is repeated below (for first, second and n^{th} order derivatives)

$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{L}} sF(s) - f(0^-)$$

$$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2 F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\frac{d^n f(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \dot{f}(0^-) - \cdots - s\overset{(n-2)}{f}(0^-) - \overset{(n-1)}{f}(0^-)$$

4. Integration

The integration theorem states that

$$\int_{0^-}^{t} f(\lambda)d\lambda \xleftrightarrow{\mathcal{L}} \frac{F(s)}{s}$$

We prove it by starting by integration by parts

$$\mathcal{L}\left(\int_{0^-}^{t} f(\lambda)d\lambda\right) = \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f(\lambda)d\lambda\right)e^{-st}dt$$

$$\int_a^b u \cdot dv = u \cdot v\Big|_a^b - \int_a^b v \cdot du$$

$$du = f(t)dt \quad u = \int_{0^-}^{t} f(\lambda)d\lambda$$

$$v = -\frac{1}{s}e^{-st} \qquad dv = e^{-st}dt$$

$$\int_{0^-}^{\infty}\left(\int_{0^-}^{t} f(\lambda)d\lambda\right)e^{-st}dt = \left[-\frac{1}{s}e^{-st}\int_{0^-}^{t} f(\lambda)d\lambda\right]_{0^-}^{\infty} - \int_{0^-}^{\infty}\left(-\frac{1}{s}e^{-st}\right)f(t)dt$$

$$= -\frac{1}{s}\left[e^{-st}\int_{0^-}^{t} f(\lambda)d\lambda\right]_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} e^{-st}f(t)dt$$

$$= -\frac{1}{s}\left[e^{-st}\int_{0^-}^{t} f(\lambda)d\lambda\right]_{0^-}^{\infty} + \frac{1}{s}F(s)$$

$$= -\frac{1}{s}\left(e^{-s\infty}\int_{0^-}^{\infty} f(\lambda)d\lambda - e^{-s0^-}\int_{0^-}^{0^-} f(\lambda)d\lambda\right) + \frac{1}{s}F(s)$$

The first term in the parentheses goes to zero if f(t) grows more slowly than an exponential (one of our requirements for existence of the Laplace Transform), and the second term goes to zero because the limits on the integral are equal. So the theorem is proven

$$\mathcal{L}\left(\int_{0^-}^{t} f(\lambda)d\lambda\right) = \frac{1}{s}F(s)$$

Example: Find Laplace Transform of Step and Ramp using Integration Property

Given that the Laplace Transform of the impulse δ(t) is Δ(s)=1, find the Laplace Transform of the step and ramp.

Solution:

We know that

$$\gamma(t) = \int_{0-}^{t} \delta(t)dt$$

so that

$$\Gamma(s) = \frac{1}{s}\Delta(s) = \frac{1}{s}$$

Likewise:

$$\mathrm{ramp}(t) = t \cdot \gamma(t) = \int_{0-}^{t} \gamma(t)dt$$

$$\mathrm{Ramp}(s) = \frac{1}{s}\Gamma(s) = \frac{1}{s^2}$$

5. Convolution

The convolution theorem states (if you haven't studied convolution, you can skip this theorem)

$$f(t) * g(t) \xleftrightarrow{\mathcal{L}} F(s) \cdot G(s)$$

note: we assume both f(t) and g(t) are *causal*.

We start our proof with the definition of the Laplace Transform

$$\mathcal{L}(f(t) * g(t)) = \int_{0^-}^{\infty} (f(t) * g(t))\, e^{-st}dt = \int_{0^-}^{\infty} \left(\int_{-\infty}^{\infty} f(\lambda)g(t-\lambda)d\lambda\right) e^{-st}dt$$

From there we continue:

$$\mathcal{L}(f(t) * g(t)) = \int_{0^-}^{\infty}\int_{-\infty}^{\infty} f(\lambda)g(t-\lambda)e^{-st}d\lambda dt$$

We can change the order of integration.

$$= \int_{-\infty}^{\infty}\int_{0^-}^{\infty} f(\lambda)g(t-\lambda)e^{-st}dtd\lambda$$

Now, we pull f(λ) out because it is constant with respect to the variable of integration, t

$$= \int_{-\infty}^{\infty} f(\lambda)\int_{0^-}^{\infty} g(t-\lambda)e^{-st}dtd\lambda$$

Now we make a change of variables

$$u = t-\lambda; \quad du = dt; \quad t = u+\lambda$$

$$\mathcal{L}(f(t) * g(t)) = \int_{-\infty}^{\infty} f(\lambda)\int_{-\lambda}^{\infty} g(u)e^{-s(u+\lambda)}dud\lambda$$

Since g(u) is zero for u<0, we can change the lower limit on the inner integral to 0^-.

$$= \int_{-\infty}^{\infty} f(\lambda)\int_{0}^{\infty} g(u)e^{-su}e^{-s\lambda}dud\lambda$$

We can pull $e^{-s\lambda}$ out (it is constant with respect to integration).

$$= \int_{-\infty}^{\infty} f(\lambda)e^{-s\lambda}\int_{0^-}^{\infty} g(u)e^{-su}dud\lambda$$

We can separate the integrals since the inner integral doesn't depend on λ.

$= \int_{-\infty}^{\infty} f(\lambda)e^{-s\lambda}d\lambda \int_{0^-}^{\infty} g(u)e^{-su}du$ We can change the lower limit on the first integral since f(λ) is causal.

$= \int_{0^-}^{\infty} f(\lambda)e^{-s\lambda}d\lambda \int_{0^-}^{\infty} g(u)e^{-su}du$ Finally we recognize that the two integrals are simply Laplace Transforms.

$\mathcal{L}\left(f(t) * g(t)\right) = F(s)G(s)$ The Theorem is proven

6. Initial Value Theorem

The initial value theorem states

$$\lim_{s\to\infty}\left(sF(s)\right) = f\left(0^+\right)$$

To show this, we first start with the Derivative Rule:

$$\frac{df(t)}{dt} \xleftrightarrow{\mathcal{L}} sF(s) - f\left(0^-\right)$$

We then invoke the definition of the Laplace Transform, and split the integral into two parts:

$$sF(s) - f\left(0^-\right) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st}dt = \int_{0^-}^{\infty} \dot{f}(t) e^{-st}dt$$

$$= \int_{0^-}^{0^+} \dot{f}(t)e^{-st}dt + \int_{0^+}^{\infty} \dot{f}(t) e^{-st}dt$$

We take the limit as s→∞:

$$\lim_{s\to\infty}\left(sF(s) - f\left(0^-\right)\right) = \lim_{s\to\infty}\left(\int_{0^-}^{0^+} \dot{f}(t)dt + \int_{0^+}^{\infty} \dot{f}(t)e^{-st}dt\right)$$

Several simplifications are in order. In the left hand expression, we can take the second term out of the limit, since it doesn't depend on 's.' In the right hand expression, we can take the first term out of the limit for the same reason, and if we substitute infinity for 's' in the second term, the exponential term goes to zero:

$$\lim_{s\to\infty}\left(sF(s)\right) - f\left(0^-\right) = \int_{0^-}^{0^+} \dot{f}(t)\,dt + \int_{0^+}^{\infty} \dot{f}(t)\,0dt$$

$$= \int_{0^-}^{0^+} \dot{f}(t)\,dt$$

$$= f\left(0^+\right) - f\left(0^-\right)$$

The two $f(0^-)$ terms cancel each other, and we are left with the Initial Value Theorem

$$\lim_{s\to\infty}\left(sF(s)\right) = f\left(0^+\right)$$

This theorem only works if F(s) is a strictly proper fraction in which the numerator polynomial is of lower order then the denominator polynomial. In other words is will work for F(s)=1/(s+1) but not F(s)=s/(s+1).

7. Final Value Theorem

The final value theorem states that if a final value of a function exists that

$$\lim_{s\to 0}\left(sF(s)\right) = \lim_{t\to\infty} f(t)$$

However, we can only use the final value if the value exists (function like sine, cosine and the ramp function don't have final values). To prove the final value theorem, we start as we did for the initial value theorem, with the Laplace Transform of the derivative,

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st}dt = sF(s) - f(0^-)$$

We let s→0,

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st}dt\right) = \lim_{s\to 0}\left(sF(s) - f(0^-)\right)$$

As s→0 the exponential term disappears from the integral. Also, we can take f(0-) out of the limit (since it doesn't depend on s)

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty} \frac{df(t)}{dt} dt\right) = \lim_{s\to 0}\left(sF(s)\right) - f(0^-)$$

We can evaluate the integral

$$\lim_{s\to 0}\left(f(\infty) - f(0^-)\right) = \lim_{s\to 0}\left(sF(s)\right) - f(0^-)$$

Neither term on the left depends on s, so we can remove the limit and simplify, resulting in the final value theorem

$$f(\infty) - \cancel{f(0^-)} = \lim_{s\to 0}\left(sF(s)\right) - \cancel{f(0^-)}$$

$$f(\infty) = \lim_{s\to 0}\left(sF(s)\right)$$

Examples of functions for which this theorem can't be used are increasing exponentials (like e^{at} where a is a positive number) that go to infinity as t increases, and oscillating functions like sine and cosine that don't have a final value.

Properties of Laplace Transform - Summary

Some of the Laplace transformation properties are:

If $f_1(t) \longleftrightarrow F_1(s)$ and [note: ⟷ implies Laplace Transform]

$f_2(t) \longleftrightarrow F_2(s)$, then

Properties Name	Illustration
Definition	$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$ $f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$

Linearity	$Af_1(t) + Bf_2(t) \overset{\mathcal{L}}{\leftrightarrow} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2f(t)}{dt^2} \overset{\mathcal{L}}{\leftrightarrow} s^2F(s) - sf(0^-) - \dot{f}(0^-)$
n^th Derivative	$\frac{d^nf(t)}{dt^n} \overset{\mathcal{L}}{\leftrightarrow} s^nF(s) - \sum_{i=1}^{n} s^{n-i}f^{(i-1)}(0^-)$
Integration	$\int_0^t f(\lambda)d\lambda \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}F(s)$
Multiplication by time	$tf(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{dF(s)}{ds}$
Time Shift	$f(t-a)\gamma(t-a) \overset{\mathcal{L}}{\leftrightarrow} e^{-as}F(s)$ ($\gamma(t)$ = unit step function)
Complex Shift	$f(t)e^{-at} \overset{\mathcal{L}}{\leftrightarrow} F(s+a)$
Time Scaling	$f\left(\frac{t}{a}\right) \overset{\mathcal{L}}{\leftrightarrow} aF(as)$
Convolution ('*' denotes convolution of functions)	$f_1(t) * f_2(t) \overset{\mathcal{L}}{\leftrightarrow} F_1(s)F_2(s)$
Initial Value Theorem (if F(s) is a strictly proper fraction)	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value Theorem (if final value exists, e.g., decaying exponentials)	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

INVERSE LAPLACE TRANSFORM

The inverse Laplace transform is represented by using the symbol **L^{-1}** it is defined as the converse of Laplace transform. If a Laplace transform is given, say F(s), the inverse of this will be L^{-1}[F(s)], this will give us the original function, i.e., f(t) (before Laplace transformation was applied to it)
Methods of finding inverse Laplace transform

- Inversion formula
- Use of tables of Laplace transform pairs
- Partial fraction expansion method

Inversion formula

This is a procedure that holds to all classes of transform function that involve the evaluation of a line integral in the complex s-plane.

$$f(t) = \int_c X(s)e^{st}\, ds$$

In the integral, the real C is to be selected such that $\sigma_1<C<\sigma_2$ when ROC of X(s) is $\sigma_1<Re\{s\}<\sigma_2$.

Use of tables of Laplace transform pairs

In this method X(s) is expressed as a sum of individual Laplace transforms,
i.e. $X(s) = X_1(s)+ X_2(s)+.... X_n(s)$

where $X_1(s), X_2(s),.... X_n(s)$ are the transforms with known inverse transforms $x_1(t), x_2(t).... x_n(t)$.

Then by linearity property, it follows that $x(t)=x_1(t)+ x_2(t)+.... +x_n(t)$.

Partial fraction expansion method

The partial fraction expansion is a practical way to invert the Laplace transform X(s)

If X(s) is a rational function of the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{k(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

Case a:

When X(s) is a proper rational function (m<n) and when all the poles of X(s) are simple or distinct, then X(s) can be written as:

$$X(s) = \frac{C_1}{(s-p_1)} + \frac{C_2}{(s-p_2)} + \cdots + \frac{C_n}{(s-p_n)}$$

Where coefficient of C_k are given by, $C_k = (s-p_k)X(s)|_{s=p_k}$

Case b:

When X(s) is a proper rational function (m<n) and if D(s) has multiple roots, i.e., if D(s) contains fractions of the form $(s-p_i)^r$,then we say that p_i is the multiple pole of X(s) with multiplicity r.

Then the expansion of X(s) consists of the terms of the form:

$$X(s) = \frac{d_1}{(s-p_i)} + \frac{d_2}{(s-p_i)^2} + \cdots + \frac{d_r}{(s-p_i)^r}$$

Where $d_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\{(s-p_i)^r X(s)\}_{s=p_i}$

Case c:

When X(s) is an improper rational function (m≥n)

Then $X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$

N(s) is the numerator polynomial in s of X(s)

D(s) is the denominator polynomial in s of X(s)

R(s) is the reminder polynomial in s with degree less than n

Q(s) is the quotient polynomial in s with degree (m-n)

Then inverse Laplace transform is computed.

Problems

1. Find the inverse Laplace transform using partial fraction expansion method

a. $X(s) = \frac{1}{(s+1)(s+2)}$

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

Putting s=-1, A=1

Putting s=-2, B=-1

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = [e^{-t} - e^{-2t}]u(t)$$

b. $X(s) = \frac{s+2}{s(s+3)}$

$$X(s) = \frac{s+2}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$s+2 = A(s+3) + Bs$$

putting s=0, A=2/3

putting s=-3, B=1/3

$$X(s) = \frac{2/3}{s} + \frac{1/3}{s+3}$$

$$x(t) = \frac{1}{3}(2 + e^{-3t})u(t)$$

c. $X(s) = \frac{1}{s(s+1)(s+2)}$

$$X(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$
$$1 = A(s+2)(s+1) + Bs(s+2) + Cs(s+1)$$

Putting s=0, A=1/2

Putting s=-1, B=-1

Putting s=-2, C=1/2

$$X(s) = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$
$$x(t) = \left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right]u(t)$$

2. Find Inverse Laplace transform of

a. $F(s) = \frac{2}{s^3}$

$$L^{-1}\left[\frac{2}{s^3}\right] = L^{-1}\left[\frac{2!}{s^3}\right] = t^2$$

b. $F(s) = \frac{2}{s^2+4}$

$$L^{-1}\left[\frac{2}{s^2+4}\right] = L^{-1}\left[\frac{2}{s^2+2^2}\right] = sin2t$$

c. $F(s) = \frac{s+1}{s^2+2s+10}$

$$L^{-1}\left[\frac{s+1}{s^2+2s+10}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+3^2}\right] = e^{-t}cos3t$$

d. $F(s) = \frac{5s}{s^2+9}$

$$L^{-1}\left[\frac{5s}{s^2+9}\right] = L^{-1}\left[\frac{5s}{s^2+3^2}\right] = 5cos3t$$

UNSOLVED PROBLEMS

1. Find the Laplace transform of
   a. $2 - 2e^{t} + 0.5\sin(4t)$
   b. $e^{-t}\sin(5t)$
   c. $e^{2t} + 2e^{-2t} - t^{2}$
2. Determine the inverse Laplace transform of
   a. $\frac{s-1}{s(s+1)}$
   b. $\frac{s^3+1}{s(s+1)(s+2)}$
   c. $\frac{s-1}{(s+1)(s^2+2s+5)}$
3. Find the initial value of continuous time signal if its Laplace transform is given as $X(s) = \frac{2s+1}{s^2-1}$

SIGNALS & SYSTEMS

Learning Material

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